

# THE GAUSS-MANIN CONNECTION AND NONCOMMUTATIVE TORI

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ABSTRACT. Given a smooth one parameter deformation of associative algebras, we define Getzler's Gauss-Manin connection on both the periodic cyclic homology and cohomology of the corresponding smooth field of algebras and investigate some basic properties. We illustrate the Gauss-Manin connection on noncommutative tori and give a deformation-theoretic calculation of the periodic cyclic cohomology of noncommutative tori.

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## 1. INTRODUCTION

In his paper [6], Getzler constructed a connection on the periodic cyclic complex of a formal deformation of an  $A_\infty$ -algebra. His connection, called the *Gauss-Manin connection*, commutes with the boundary map on the periodic cyclic complex and descends to a flat connection on the periodic cyclic homology of the deformation.

Our goal is to define this connection for a more analytic, and less algebraic, class of deformations and investigate its properties. The type of deformation we consider is as follows. Let  $X$  be a Fréchet space and let  $\{m_t\}_{t \in J}$  be a family of continuous associative multiplications on  $X$  which are parametrized by a single real parameter  $t \in J \subset \mathbb{R}$ , and which vary smoothly in some sense. Thus, for each  $t \in J$ , we have a Fréchet algebra  $A_t$  whose underlying space is  $X$  and whose multiplication is given by  $m_t$ . These algebras can be collected to form the algebra  $A$  of sections of the vector bundle over  $J$  whose fiber at  $t \in J$  is  $A_t$ , where the multiplication is defined fiberwise. The section algebra  $A$  is an algebra over  $C^\infty(J)$ , the space

of smooth complex-valued functions defined on the parameter space  $J$ , where the module action is given by fiberwise scalar multiplication.

The complex of interest to us is the periodic cyclic complex of  $A$  over the ground ring  $C^\infty(J)$ . Like the section algebra, this can be thought of as a bundle of complexes over  $J$  whose fiber at  $t \in J$  is the periodic cyclic complex of  $A_t$ . It is on this complex that we shall define Getzler's Gauss-Manin connection  $\nabla_{GM}$ . We shall investigate some basic general properties of  $\nabla_{GM}$ . The most notable perhaps is the fact that

$$\nabla_{GM}[\text{ch } P] = 0,$$

where  $[\text{ch } P]$  denotes the class in  $HP_0(A)$  of the Chern character of an idempotent  $P \in M_N(A)$ .

There is also a dual connection  $\nabla^{GM}$  on the periodic cyclic cohomology  $HP^\bullet(A)$  over  $C^\infty(J)$ . The connections are compatible in the sense that for  $[\varphi] \in HP^\bullet(A)$  and  $[\omega] \in HP_\bullet(A)$ ,

$$\frac{d}{dt}\langle \varphi, \omega \rangle = \langle \nabla^{GM} \varphi, \omega \rangle + \langle \varphi, \nabla_{GM} \omega \rangle,$$

where

$$\langle \cdot, \cdot \rangle : HP^\bullet(A) \times HP_\bullet(A) \rightarrow C^\infty(J)$$

is the canonical pairing. Combined with the above result, this says

$$\frac{d}{dt}\langle \varphi, \text{ch } P \rangle = \langle \nabla^{GM} \varphi, \text{ch } P \rangle$$

for an idempotent  $P \in M_N(A)$ . This offers insight into how the pairing between K-theory and cyclic cohomology deforms as the algebra deforms, as we will see with some calculations in noncommutative tori.

The fundamental issue for us is to determine when we can parallel transport with respect to  $\nabla_{GM}$  at the level of periodic cyclic homology. Indeed, doing so would provide isomorphisms  $HP_\bullet(A_t) \cong HP_\bullet(A_s)$  between the periodic cyclic homologies of any two algebras in the deformation. This can be used as a computational device if one already knows the cyclic homology of one particular algebra  $A_{t_0}$  in the deformation. Of course, the striking degree of generality for which  $\nabla_{GM}$  exists is an indication that any attempt to integrate  $\nabla_{GM}$  will fail generally. The goal is then to identify properties of a deformation that allow for parallel transport.

The main examples for us are the (smooth) noncommutative tori. An  $n$ -dimensional noncommutative torus  $\mathcal{A}_\Theta$  can be viewed as a deformation of the algebra  $C^\infty(\mathbb{T}^n)$  of smooth functions on the  $n$ -torus. The cyclic cohomologies of noncommutative tori were first calculated by Connes [3] in the case  $n = 2$ , and later by Nest [11] in the general case. It was also shown in [3] that the even (resp. odd) continuous periodic cyclic homology of a smooth manifold  $M$  is isomorphic to the direct sum of its even (resp. odd) de Rham cohomology groups. Our main result is that parallel transport isomorphisms exist for a deformation of noncommutative tori, and in particular  $HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet(C^\infty(\mathbb{T}^n))$ . Combining this with Connes' result for smooth manifolds gives a deformation theoretic calculation of  $HP_\bullet(\mathcal{A}_\Theta)$ .

The paper is organized as follows. In §2, we review necessary preliminary functional analytic definitions and results, including locally convex algebras, locally convex modules, and projective tensor products over locally convex algebras. In §3, we introduce the class of deformations that we shall work with. §4 contains a review of the Hochschild and cyclic homology of an algebra, and §5 contains an account of

operations on the cyclic complex and the Cartan homotopy formula of Getzler [6]. §6 deals with connections on a module and the notion of parallel transport along a connection. In §7, we define Getzler's Gauss-Manin connection in our context of smooth deformations. The remainder of the paper is devoted to the deformation of noncommutative tori. In §8, we introduce a variant of cyclic homology of an algebra with an action of a Lie algebra  $\mathfrak{g}$ . The corresponding complex is easier to work with than the ordinary cyclic complex in the case of noncommutative tori. In §9, we give deformation-theoretic proof that the periodic cyclic cohomology of  $\mathcal{A}_\Theta$  is independent of  $\Theta$ . Finally, in §10, we use our calculations with the Gauss-Manin connection to draw conclusions about the pairing between K-theory and cyclic cohomology for the noncommutative tori deformation.

For the sake of presentation, some proofs from §§2-3 are postponed to an appendix.

## 2. FUNCTIONAL ANALYTIC PRELIMINARIES

All locally convex topological vector spaces that follow are assumed to be Hausdorff and defined over the ground field  $\mathbb{C}$ .

**2.1. Locally convex algebras, locally convex modules, and tensor products.** Given two locally convex vector spaces  $X$  and  $Y$ , the *projective topology* on  $X \otimes Y$  is the strongest locally convex topology such that the canonical bilinear map  $\iota : X \times Y \rightarrow X \otimes Y$  is jointly continuous, see [15, Chapter 43]. The (*completed*) *projective tensor product*  $X \hat{\otimes} Y$  is the completion of  $X \otimes Y$  with the projective topology. The completed projective tensor product has the universal property that any jointly continuous bilinear map  $B$  from  $X \times Y$  into a complete locally convex space  $Z$  induces a unique continuous linear map  $\hat{B} : X \hat{\otimes} Y \rightarrow Z$  such that the diagram

$$\begin{array}{ccc} X \times Y & & \\ \downarrow \iota & \xrightarrow{B} & \\ X \hat{\otimes} Y & \xrightarrow{\hat{B}} & Z \end{array}$$

commutes. The projective tensor product is functorial in the sense that two continuous linear maps  $F : X_1 \rightarrow X_2$  and  $G : Y_1 \rightarrow Y_2$  induce a continuous linear map

$$F \otimes G : X_1 \hat{\otimes} Y_1 \rightarrow X_2 \hat{\otimes} Y_2$$

given on elementary tensors by

$$(F \otimes G)(x \otimes y) = F(x) \otimes G(y).$$

By a *locally convex algebra*, we mean a locally convex space  $A$  equipped with a jointly continuous associative multiplication. We shall only be interested in the case where  $A$  is complete, and in this case multiplication gives a continuous linear map  $m : A \hat{\otimes} A \rightarrow A$ . Joint continuity implies that for every defining seminorm  $p$  on  $A$ , there is another continuous seminorm  $q$  such that

$$p(ab) \leq q(a)q(b), \quad \forall a, b \in A.$$

There may be no relationship between  $p$  and  $q$  in general. In the special case where

$$p(ab) \leq p(a)p(b), \quad \forall a, b \in A,$$

for a family of seminorms defining the topology, we say that the algebra is *multiplicatively convex* or *m-convex*.

Now suppose  $R$  is a unital commutative locally convex algebra. By a *locally convex  $R$ -module*, we mean an  $R$ -module  $M$  with a locally convex topology for which the module action is jointly continuous. All modules will be assumed to be unital in the sense that  $1 \cdot m = m$  for all  $m \in M$ . A *locally convex  $R$ -algebra* is an  $R$ -algebra  $A$  which is simultaneously a locally convex algebra and a locally convex  $R$ -module.

We shall often have to take topological tensor products over an algebra different from  $\mathbb{C}$ . The basic facts we need are below, but a more detailed exposition can be found in [7, Chapter II]. Suppose  $R$  is a unital commutative locally convex algebra and let  $M$  and  $N$  be locally convex  $R$ -modules. The *(completed) projective tensor product over  $R$*  of  $M$  and  $N$ , denoted  $M \widehat{\otimes}_R N$ , is defined to be the completion<sup>1</sup> of  $(M \widehat{\otimes}_{\mathbb{C}} N)/K$  where  $K$  is the closure of the subspace spanned by elements of the form

$$(r \cdot m) \otimes n - m \otimes (r \cdot n), \quad r \in R, m \in M, n \in N.$$

Given an elementary tensor  $m \otimes n \in M \widehat{\otimes}_{\mathbb{C}} N$ , we shall denote its image in  $M \widehat{\otimes}_R N$  again by  $m \otimes n$ . Then  $M \widehat{\otimes}_R N$  is a locally convex  $R$ -module under the action

$$r \cdot (m \otimes n) := (r \cdot m) \otimes n = m \otimes (r \cdot n),$$

and the canonical map  $\iota : M \times N \rightarrow M \widehat{\otimes}_R N$  is  $R$ -bilinear. The completed projective tensor product over  $R$  has the universal property that any jointly continuous  $R$ -bilinear map  $B$  from  $M \times N$  into a complete locally convex  $R$ -module  $P$  induces a unique continuous  $R$ -linear map  $\widehat{B} : M \widehat{\otimes}_R N \rightarrow P$  such that the diagram

$$\begin{array}{ccc} M \times N & & \\ \downarrow \iota & B & \\ M \widehat{\otimes}_R N & \xrightarrow{\widehat{B}} & P \end{array}$$

commutes. The projective tensor product also has the functorial property that two continuous  $R$ -linear maps  $F : M_1 \rightarrow N_1$  and  $G : M_2 \rightarrow N_2$  induce a continuous  $R$ -linear map

$$F \otimes G : M_1 \widehat{\otimes}_R N_1 \rightarrow M_2 \widehat{\otimes}_R N_2$$

in the usual way.

A locally convex  $R$ -module is *free* if it is isomorphic to  $R \widehat{\otimes}_{\mathbb{C}} X$  for some locally convex space  $X$ . Here the  $R$ -module action is given by

$$r \cdot (s \otimes x) = rs \otimes x.$$

Given a locally convex  $R$ -module  $M$ , the familiar algebraic isomorphism

$$\text{Hom}(X, M) \cong \text{Hom}_R(R \otimes X, M)$$

holds true in the locally convex category.

**Proposition 2.1.** *Given a locally convex space  $X$  and a complete locally convex  $R$ -module  $M$ , there is a one-to-one correspondence between the set  $\text{Hom}(X, M)$  of continuous linear maps from  $X$  to  $M$  and the set  $\text{Hom}_R(R \widehat{\otimes}_{\mathbb{C}} X, M)$  of continuous  $R$ -linear maps from  $R \widehat{\otimes}_{\mathbb{C}} X$  to  $M$ .*

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<sup>1</sup>If  $M$  and  $N$  are Fréchet spaces, then  $(M \widehat{\otimes}_{\mathbb{C}} N)/K$  is already complete.

**Proposition 2.2.** *Given two locally convex spaces  $X$  and  $Y$ ,*

$$(R\widehat{\otimes}_{\mathbb{C}}X)\widehat{\otimes}_R(R\widehat{\otimes}_{\mathbb{C}}Y) \cong R\widehat{\otimes}_{\mathbb{C}}(X\widehat{\otimes}_{\mathbb{C}}Y)$$

*as locally convex  $R$ -modules via the correspondence*

$$(r_1 \otimes x) \otimes (r_2 \otimes y) \leftrightarrow r_1 r_2 \otimes (x \otimes y).$$

Hence the projective tensor product of free modules is free.

We shall largely be concerned with algebras and modules for which the underlying locally convex space is a Fréchet space. We shall call them *Fréchet algebras*<sup>2</sup> and *Fréchet  $R$ -modules* accordingly. If  $M$  and  $N$  are Fréchet  $R$ -modules, then  $M\widehat{\otimes}_R N$  is a Fréchet  $R$ -module.

**2.2. The space  $C^\infty(J, X)$ .** Let  $J$  be a nonempty open interval of real numbers and let  $X$  be a locally convex vector space. Consider the space  $C^\infty(J, X)$  of infinitely differentiable functions on  $J$  with values in  $X$ . We equip  $C^\infty(J, X)$  with its usual topology of uniform convergence of functions and all their derivatives on compact subsets of  $J$ . We shall write  $C^\infty(J) = C^\infty(J, \mathbb{C})$ , which is a Fréchet algebra under this topology and  $C^\infty(J, X)$  is a locally convex module over  $C^\infty(J)$ . If  $X$  is complete, then

$$C^\infty(J, X) \cong C^\infty(J)\widehat{\otimes}X,$$

that is,  $C^\infty(J, X)$  is a free locally convex  $C^\infty(J)$ -module, see e.g. [15, Theorem 44.1]. In particular, if  $X$  is a Fréchet space, then  $C^\infty(J, X)$  is a Fréchet space.

Given  $x \in C^\infty(J, X)$ , we shall denote by  $x^{(n)}$  the  $n$ -th derivative of  $x$ , which again is an element of  $C^\infty(J, X)$ . The space  $C^\infty(J, X)$  is equipped with continuous linear “evaluation maps”

$$\epsilon_t : C^\infty(J, X) \rightarrow X$$

for each  $t \in J$  given by  $\epsilon_t(x) = x(t)$ .

**Definition 2.3.** Given two locally convex spaces  $X$  and  $Y$ , a *smooth family of continuous linear maps* from  $X$  to  $Y$  is a collection of continuous linear maps

$$F_t : X \rightarrow Y, \quad t \in J$$

which vary smoothly in the sense that the formula

$$F(x)(t) = F_t(x)$$

defines a continuous map

$$F : X \rightarrow C^\infty(J, Y).$$

The adjunction

$$\mathrm{Hom}(X, C^\infty(J, Y)) \cong \mathrm{Hom}_{C^\infty(J)}(C^\infty(J)\widehat{\otimes}X, C^\infty(J, Y))$$

of Proposition 2.1 implies the following result.

**Proposition 2.4.** *If  $X$  and  $Y$  are complete locally convex vector spaces, then there is a one-to-one correspondence between the set of smooth families of continuous linear maps from  $X$  to  $Y$  and the set of continuous  $C^\infty(J)$ -linear maps from  $C^\infty(J, X)$  to  $C^\infty(J, Y)$ .*

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<sup>2</sup>We do not insist that our Fréchet algebras are  $m$ -convex, as some authors do.

The correspondence is as follows. To a smooth family  $\{F_t\}$ , we associate  $F : C^\infty(J, X) \rightarrow C^\infty(J, Y)$  given by

$$F(x)(t) = F_t(x(t)).$$

Conversely, given  $F : C^\infty(J, X) \rightarrow C^\infty(J, Y)$ , we have the family  $\{F_t\}$  where

$$F_t(x) = F(x)(t).$$

Here  $x \in X$  is viewed as a “constant function” in  $C^\infty(J, X)$ .

Given a smooth family  $\{F_t : X \rightarrow Y\}_{t \in J}$  of continuous linear maps, it is necessary that the map

$$t \mapsto F_t(x)$$

is smooth for each  $x \in X$ . If  $X$  is barreled, e.g. if  $X$  is Fréchet, this is also sufficient.

**Proposition 2.5.** *If  $X$  is barreled, then  $\{F_t : X \rightarrow Y\}_{t \in J}$  is a smooth family of continuous linear maps if and only if the map*

$$t \mapsto F_t(x)$$

*is smooth for every  $x \in X$ .*

### 3. ONE-PARAMETER DEFORMATIONS OF AN ALGEBRA

Let  $X$  be a complete, locally convex vector space and let  $J$  denote an open interval of real numbers.

**Definition 3.1.** A *smooth one-parameter deformation* is a smooth family of continuous linear maps  $\{m_t : X \hat{\otimes} X \rightarrow X\}_{t \in J}$  for which each  $m_t$  is associative.

So for each  $t \in J$ , we have a locally convex algebra  $A_t := (X, m_t)$  whose underlying space is  $X$ . If all the algebras  $A_t$  are unital, then we say the deformation is *unit-preserving* if there is an element  $1 \in X$  such that

$$m_t(1, a) = m_t(a, 1) = a, \quad \forall a \in X, t \in J.$$

All deformations that follow are assumed to be unit-preserving.

Given such a deformation  $\{m_t\}_{t \in J}$  on  $X$ , consider the continuous  $C^\infty(J)$ -linear map

$$m : C^\infty(J, X \hat{\otimes} X) \rightarrow C^\infty(J, X)$$

associated to  $\{m_t\}$  as in Proposition 2.4. If  $A = C^\infty(J, X)$ , then  $m$  can be viewed as a map

$$m : A \hat{\otimes}_{C^\infty(J)} A \rightarrow A$$

using Proposition 2.2. Associativity of  $m$  follows from associativity of the family  $\{m_t\}$ . Thus  $A$  is a locally convex  $C^\infty(J)$ -algebra, which we shall refer to as the *section algebra* of the deformation  $\{A_t\}_{t \in J}$ . Explicitly, the multiplication in  $A$  is given by

$$(a_1 a_2)(t) = m_t(a_1(t), a_2(t)).$$

Note that the “evaluation maps”  $\epsilon_t : A \rightarrow A_t$  are algebra maps.

**Proposition 3.2.** *Associating to a deformation its section algebra gives a one-to-one correspondence between smooth one-parameter deformations over  $J$  with underlying space  $X$  and locally convex  $C^\infty(J)$ -algebra structures on  $C^\infty(J, X)$ .*

**Proposition 3.3.** *If  $X$  is a Fréchet space, then a set of continuous associative multiplications  $\{m_t : X \widehat{\otimes} X \rightarrow X\}_{t \in J}$  is a smooth one-parameter deformation if and only if the map*

$$t \mapsto m_t(x_1, x_2)$$

*is smooth for each fixed  $x_1, x_2 \in X$ .*

Now, suppose  $A$  and  $B$  are the section algebras of two smooth one-parameter deformations  $\{A_t\}_{t \in J}$  and  $\{B_t\}_{t \in J}$  with underlying spaces  $X$  and  $Y$  respectively.

**Definition 3.4.** A *morphism*  $\{F_t\}_{t \in J}$  between two deformations  $\{A_t\}_{t \in J}$  and  $\{B_t\}_{t \in J}$  is a smooth family  $\{F_t\}_{t \in J}$  of algebra homomorphisms. That is,

- (i) for each  $t \in J$ ,  $F_t$  is an algebra map with respect to the algebra structures of  $A_t$  and  $B_t$ .
- (ii) the family  $\{F_t\}_{t \in J}$  is smooth in the sense of Definition 2.3.

**Proposition 3.5.** *There is a one-to-one correspondence between morphisms from  $\{A_t\}_{t \in J}$  to  $\{B_t\}_{t \in J}$  and continuous  $C^\infty(J)$ -linear algebra maps from  $A$  to  $B$ .*

*Proof.* Apply Proposition 2.4.  $\square$

**3.1. Smooth noncommutative tori.** Given an  $n \times n$  skew-symmetric real valued matrix  $\Theta$ , the *noncommutative torus*  $A_\Theta$  is the universal  $C^*$ -algebra generated by  $n$  unitaries  $u_1, \dots, u_n$  such that

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j,$$

where  $\Theta = (\theta_{jk})$ . In the case  $\Theta = 0$ , all of the generating unitaries commute, and we have  $A_0 \cong C(\mathbb{T}^n)$ , the algebra of continuous functions on the  $n$ -torus. It is for this reason why the algebra  $A_\Theta$  has earned its name, as it can be philosophically viewed as functions on some “noncommutative torus” in the spirit of Alain Connes’ noncommutative geometry [4]. We shall be interested in a dense subalgebra  $\mathcal{A}_\Theta \subset A_\Theta$  which plays the role of the smooth functions on the noncommutative torus. Indeed, for the case  $\theta = 0$ ,  $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$ , the algebra of smooth functions on  $\mathbb{T}^n$ . As a topological vector space,  $\mathcal{A}_\Theta$  is the Schwarz space  $\mathcal{S}(\mathbb{Z}^n)$  of complex-valued sequences indexed by  $\mathbb{Z}^n$  of rapid decay, defined as follows. Given a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ , we shall write

$$|\alpha| = |\alpha_1| + \dots + |\alpha_n|.$$

A sequence  $x = (x_\alpha)_{\alpha \in \mathbb{Z}^n}$  is of *rapid decay* if for every positive integer  $k$ ,

$$p_k(x) := \sum_{\alpha \in \mathbb{Z}^n} (1 + |\alpha|)^k |x_\alpha| < \infty.$$

The functions  $p_k$  are seminorms, and the topology on  $\mathcal{A}_\Theta$  is the locally convex topology defined by these seminorms. Under this topology,  $\mathcal{A}_\Theta$  is complete, and therefore is a Fréchet space. The identification of  $\mathcal{A}_\Theta$  as a subalgebra of  $A_\Theta$  is given by the map  $\iota : \mathcal{A}_\Theta \rightarrow A_\Theta$ ,

$$\iota(x) = \sum_{\alpha \in \mathbb{Z}^n} x_\alpha u^\alpha,$$

where  $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}$ . Since  $u^\alpha$  is a unitary, it has norm 1 in the  $C^*$ -algebra  $A_\Theta$ , and consequently

$$\|\iota(x)\| \leq \sum_{\alpha \in \mathbb{Z}^n} |x_\alpha| = p_0(x) < \infty.$$

This show that the series defining  $\iota(x)$  is absolutely convergent, and also that the inclusion  $\iota$  is continuous. It is clear that  $\mathcal{A}_\Theta$  is a norm dense subalgebra of  $A_\Theta$  because it contains the  $*$ -algebra generated by  $u_1, \dots, u_n$ . The multiplication is therefore given by the twisted convolution product

$$(xy)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha - \beta, \beta)} x_{\alpha - \beta} y_\beta,$$

where

$$B_\Theta(\alpha, \beta) = \sum_{j > k} \alpha_j \beta_k \theta_{jk}.$$

One can show that for any  $k$ ,

$$p_k(xy) \leq p_k(x)p_k(y), \quad \forall x, y \in \mathcal{A}_\Theta,$$

and thus  $\mathcal{A}_\Theta$  is an  $m$ -convex Fréchet algebra.

The algebra  $\mathcal{A}_\Theta$  possesses  $n$  canonical continuous derivations

$$\delta_1, \dots, \delta_n : \mathcal{A}_\Theta \rightarrow \mathcal{A}_\Theta$$

defined by

$$(\delta_j(x))_\alpha = \alpha_j \cdot x_\alpha.$$

In the case  $\theta = 0$ , the derivations  $\delta_1, \dots, \delta_n$  correspond to the usual partial differential operators up to a scalar multiple.

There is also a canonical continuous trace  $\tau : \mathcal{A}_\Theta \rightarrow \mathbb{C}$  given by  $\tau(x) = x_0$ , which corresponds to integration with respect to the normalized Haar measure in the case  $\Theta = 0$ .

The smooth noncommutative torus  $\mathcal{A}_\Theta$  can be viewed as a smooth one-parameter deformation of  $C^\infty(\mathbb{T}^n) \cong \mathcal{A}_0$  in the following way. For each  $t \in J = \mathbb{R}$ , let  $A_t = \mathcal{A}_{t\Theta}$ . The product in  $A_t$  is given by

$$m_t(x, y)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha - \beta, \beta)t} x_{\alpha - \beta} y_\beta.$$

**Proposition 3.6.** *Given an  $n \times n$  skew-symmetric real matrix  $\Theta$ , the deformation  $\{\mathcal{A}_{t\Theta}\}_{t \in \mathbb{R}}$  is smooth, and for  $x, y$  in the underlying space  $\mathcal{S}(\mathbb{Z}^n)$ ,*

$$\frac{d}{dt} m_t(x, y) = 2\pi i \sum_{j > k} \theta_{jk} m_t(\delta_j(x), \delta_k(y)).$$

#### 4. HOCHSCHILD AND CYCLIC HOMOLOGY

Let  $R$  be a complete unital commutative locally convex algebra and let  $A$  be a complete unital locally convex  $R$ -algebra. All completed projective tensor products that follow will be taken over  $R$ . The main examples for us will be  $R = \mathbb{C}$  and  $R = C^\infty(J)$ , the smooth functions over a real interval  $J$ . All homology theories that follow are the continuous versions of the usual algebraic theories, in that they take into account the topology of the algebra  $A$ .



**4.1. Hochschild cochains.** Let  $C^n(A, A)$  denote the space of all continuous  $n$ -multilinear (over  $R$ ) maps  $D : A^{\times n} \rightarrow A$  such that  $D$  vanishes whenever 1 is one of its arguments. We equip  $C^n(A, A)$  with the topology of uniform convergence on bounded subsets. The coboundary map  $\delta : C^n(A, A) \rightarrow C^{n+1}(A, A)$  is given by

$$\begin{aligned} \delta D(a_1, \dots, a_{n+1}) &= D(a_1, \dots, a_n)a_{n+1} + (-1)^{n+1}a_1D(a_2, \dots, a_{n+1}) \\ &\quad + \sum_{j=1}^n (-1)^{n-j+1}D(a_1, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_{n+1}). \end{aligned}$$

The cohomology of  $(C^\bullet(A, A), \delta)$  is the Hochschild cohomology of  $A$  with coefficients in  $A$ , and is denoted by  $H^\bullet(A, A)$ . If we wish to emphasize the ground ring  $R$ , we shall write  $H_R^\bullet(A, A)$ .

An important example for us is that a cochain  $D \in C^1(A, A)$  satisfies  $\delta D = 0$  if and only if  $D$  is a derivation, i.e.

$$D(a_1 a_2) = D(a_1)a_2 + a_1 D(a_2) \quad \forall a_1, a_2 \in A.$$

There is an associative product

$$\smile : C^k(A, A) \otimes C^l(A, A) \rightarrow C^{k+l}(A, A)$$

given by

$$(D \smile E)(a_1, \dots, a_{k+l}) = D(a_1, \dots, a_k)E(a_{k+1}, \dots, a_{k+l}).$$

This product satisfies

$$\delta(D \smile E) = \delta D \smile E + (-1)^k D \smile \delta E,$$

so that  $(C^\bullet(A, A), \delta, \smile)$  is a differential graded algebra. It is shown in [5] that  $H^\bullet(A, A)$  is a graded commutative algebra with respect to this product. However, the product  $\smile$  is not graded commutative in  $C^\bullet(A, A)$ .

The complex  $C^\bullet(A, A)$  also admits the structure of a differential graded Lie algebra, after a degree shift [5]. Let  $\mathfrak{g}^\bullet(A) = C^{\bullet+1}(A)$ . We let  $|D|$  denote the degree of an element of  $\mathfrak{g}^\bullet(A)$ , so that if  $D$  is a  $k$ -cochain, then  $|D| = k - 1$ . Given  $D, E \in \mathfrak{g}^\bullet(A)$ , let

$$\begin{aligned} (D \circ E)(a_1, \dots, a_{|D|+|E|+1}) \\ = \sum_{i=0}^{|D|} (-1)^{i|E|} D(a_1, \dots, a_i, E(a_{i+1}, \dots, a_{i+|E|+1}), a_{i+|E|+2}, \dots, a_{|D|+|E|+1}). \end{aligned}$$

Note that  $|D \circ E| = |D| + |E|$ . The *Gerstenhaber bracket* is defined as

$$[D, E] = D \circ E - (-1)^{|D||E|} E \circ D.$$

One can check that  $(\mathfrak{g}^\bullet(A), [\cdot, \cdot])$  is a graded Lie algebra and moreover,

$$\delta[D, E] = [\delta D, E] + (-1)^{|D|}[D, \delta E],$$

so that  $\mathfrak{g}^\bullet(A)$  is a differential graded Lie algebra.

Let  $m$  denote the multiplication map for  $A$ . Strictly speaking,  $m \notin C^2(A, A)$  because we are dealing with normalized cochains. However, one can still take brackets with  $m$  using the above formulas. The equation  $[m, m] = 0$  says that  $m$  is associative. The graded Jacobi identity then implies that taking a bracket with  $m$  is a differential, and indeed it is a fact that

$$\delta D = [m, D], \quad \forall D \in \mathfrak{g}^\bullet(A).$$

That  $\delta$  is a graded Lie algebra derivation also follows from the graded Jacobi identity.

**4.2. Hochschild homology.** For  $n \geq 0$ , the Fréchet space of normalized Hochschild  $n$ -chains, denoted  $C_n(A)$ , is the quotient of  $A^{\widehat{\otimes}(n+1)}$  by the closed submodule generated by degenerate chains, that is elementary tensors  $a_0 \otimes a_1 \otimes \dots \otimes a_n$  where  $a_j \in A$  and  $a_j = 1$  for some  $1 \leq j \leq n$ . The boundary map  $b : C_n(A) \rightarrow C_{n-1}(A)$  is given on elementary tensors by

$$b(a_0 \otimes \dots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \dots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \dots \otimes a_n \\ + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}.$$

More formally,  $b$  is induced by the universal property of the projective tensor product as

$$b = \sum_{j=0}^{n-1} (-1)^j 1^{\otimes j} \otimes m \otimes 1^{\otimes(n-j-1)} + (-1)^n (m \otimes 1^{\otimes(n-1)}) \circ t,$$

where  $m : A \widehat{\otimes} A \rightarrow A$  is multiplication and  $t : A^{\widehat{\otimes}(n+1)} \rightarrow A^{\widehat{\otimes}(n+1)}$  is the cyclic permutation

$$t(a_0 \otimes \dots \otimes a_n) = a_n \otimes a_0 \otimes \dots \otimes a_{n-1}.$$

This shows that  $b$  is continuous. One can check that  $b^2 = 0$  and  $b$  maps degenerate chains to degenerate chains. The homology of the complex  $(C_\bullet(A), b)$  is called the *Hochschild homology* of  $A$  (with coefficients in  $A$ ) and shall be denoted  $H_\bullet(A)$  or  $H_\bullet^R(A)$  if we wish to emphasize  $R$ .

**4.3. Cyclic homology.** We only introduce the periodic cyclic theory. Let

$$C_{\text{even}}(A) = \prod_{n=0}^{\infty} C_{2n}(A), \quad C_{\text{odd}}(A) = \prod_{n=0}^{\infty} C_{2n+1}(A),$$

with the product topologies. Consider the operator  $B : C_n(A) \rightarrow C_{n+1}(A)$  given on elementary tensors by

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{j=0}^n (-1)^{jn} 1 \otimes a_j \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{j-1}.$$

Due to the definition of  $C_n(A)$ , it is clear that  $B^2 = 0$ . Moreover, one can check that

$$bB + Bb = 0.$$

Extend the operators  $b$  and  $B$  to the periodic cyclic complex

$$C_{\text{per}}(A) = C_{\text{even}}(A) \oplus C_{\text{odd}}(A).$$

This is a  $\mathbb{Z}/2$ -graded complex

$$C_{\text{even}}(A) \rightleftharpoons C_{\text{odd}}(A)$$

with differential  $b+B$ . The homology groups of this complex are called the even and odd *periodic cyclic homology groups* of  $A$ , and are denoted  $HP_0(A)$  and  $HP_1(A)$  respectively. As before, we will write  $HP_\bullet^R(A)$  if we wish to emphasize the ground ring  $R$ .

**4.4. Chern character.** Given an idempotent  $P \in A$ ,  $P^2 = P$ , we define  $\text{ch } P \in C_{\text{even}}(A)$  by  $(\text{ch } P)_0 = P$  and for  $n \geq 1$ ,

$$(\text{ch } P)_{2n} = (-1)^n \frac{(2n)!}{n!} (P^{\otimes(2n+1)} - \frac{1}{2} \otimes P^{\otimes(2n)})$$

Then a quick calculation shows that

$$b(\text{ch } P_{2(n+1)}) = -B(\text{ch } P_{2n})$$

and this implies that  $(b + B) \text{ch } P = 0$ .

More generally, we can define  $\text{ch } P \in C_{\text{even}}(A)$  when  $P$  is an idempotent in the matrix algebra  $M_N(A) \cong M_N(\mathbb{C}) \otimes A$ . Consider the *generalized trace*  $T : C_{\bullet}(M_N(A)) \rightarrow C_{\bullet}(A)$  defined by

$$T((u_0 \otimes a_0) \otimes \dots \otimes (u_n \otimes a_n)) = \text{tr}(u_0 \dots u_n) a_0 \otimes \dots \otimes a_n,$$

where  $\text{tr} : M_N(\mathbb{C}) \rightarrow \mathbb{C}$  is the ordinary trace. As shown in [10],  $T$  is a chain homotopy equivalence, and so induces an isomorphism  $HP_{\bullet}(M_N(A)) \cong HP_{\bullet}(A)$ . So we define  $\text{ch } P \in C_{\text{even}}(A)$  to be the image of  $\text{ch } P \in C_{\text{even}}(M_N(A))$  under the map  $T$ .

Given an invertible  $U \in A$ , we shall also construct a cycle  $\text{ch } U \in C_{\text{odd}}(A)$ . For  $n \geq 0$ , let

$$(\text{ch } U)_{2n+1} = (-1)^n n! U^{-1} \otimes U \otimes U^{-1} \otimes \dots \otimes U^{-1} \otimes U.$$

Then, one can check that

$$b(\text{ch } U_{2n+1}) = -B(\text{ch } U_{2n-1}),$$

so that  $(b + B) \text{ch } U = 0$ . As in the case of idempotents, we can define  $\text{ch } U \in C_{\text{odd}}(A)$  for any invertible  $U \in M_N(A)$  by composing with  $T$ .

**4.5. Dual cohomology theories.** To obtain periodic cyclic cohomology, we dualize the previous notions. Let  $C^n(A) = C_n(A)^* = \text{Hom}_R(C_n(A), R)$  with the topology of uniform convergence on bounded subsets. This is the space of continuous  $(n+1)$ -multilinear (over  $R$ ) maps

$$\varphi : A^{\times(n+1)} \rightarrow R$$

such that  $\varphi$  vanishes whenever 1 is placed in any of its arguments except possibly the first. The maps

$$b : C^n(A) \rightarrow C^{n+1}(A), \quad B : C^n(A) \rightarrow C^{n-1}(A)$$

are given by

$$\begin{aligned} b\varphi(a_0, \dots, a_n) &= \sum_{j=0}^{n-1} (-1)^j \varphi(a_0, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_n) \\ &\quad + (-1)^n \varphi(a_n a_0, a_1, \dots, a_{n-1}), \\ B\varphi(a_0, \dots, a_{n-1}) &= \sum_{j=0}^{n-1} (-1)^{j(n-1)} \varphi(1, a_j, \dots, a_{n-1}, a_0, \dots, a_{j-1}). \end{aligned}$$

The cohomology of  $(C^{\bullet}(A), b)$  is called the *Hochschild cohomology* of  $A$  (with coefficients in  $A^* = \text{Hom}_R(A, R)$ ) and will be denoted by  $H^{\bullet}(A)$ . The periodic cyclic

cochain complex is  $C^{\text{per}}(A) = C^{\text{even}}(A) \oplus C^{\text{odd}}(A)$ , where

$$C^{\text{even}}(A) = \bigoplus_{n=0}^{\infty} C^{2n}(A), \quad C^{\text{odd}}(A) = \bigoplus_{n=0}^{\infty} C^{2n+1}(A).$$

Then  $C^{\text{per}}(A)$  is a  $\mathbb{Z}/2$ -graded complex with differential  $b + B$ , and its cohomology groups are the even and odd *periodic cyclic cohomology* of  $A$ , denoted  $HP^0(A)$  and  $HP^1(A)$  respectively.

Since  $C^{\text{per}}(A) \cong C_{\text{per}}(A)^*$ , there is a canonical pairing

$$\langle \cdot, \cdot \rangle : C^{\text{per}}(A) \times C_{\text{per}}(A) \rightarrow R$$

which descends to a bilinear map

$$\langle \cdot, \cdot \rangle : HP^\bullet(A) \times HP_\bullet(A) \rightarrow R.$$

**4.6. Noncommutative geometry dictionary.** In the case where  $A = C^\infty(M)$ , the algebra of smooth functions on a closed manifold  $M$  with its usual Fréchet topology, the above homology groups have geometric interpretations. The Hochschild cohomology  $H^\bullet(A, A)$  is the graded space of multivector fields on  $M$ . The cup product corresponds to the wedge product of multivector fields, and the Gerstenhaber bracket corresponds to the Schouten-Nijenhuis bracket. The Hochschild homology  $H_\bullet(A)$  is the space of differential forms on  $M$ . The differential  $B$  descends to a differential on  $H_\bullet(A)$ , and this can be identified with the de Rham differential  $d$  up to a constant. The even (respectively odd) periodic cyclic homology can be identified with the direct sum of the even (respectively odd) de Rham cohomology groups. In a dual fashion, the Hochschild cohomology  $H^\bullet(A)$  is the space of de Rham currents and the periodic cyclic cohomology can be identified with de Rham homology.

Thus, for any, not necessarily commutative, algebra  $A$ , we can view  $C^\bullet(A, A)$  and  $C_\bullet(A)$  as spaces of noncommutative multivector fields and noncommutative differential forms respectively. Just as multivector fields act on differential forms by Lie derivative and contraction operations, there are Lie derivative and contraction operations

$$L, \iota : C^\bullet(A, A) \rightarrow \text{End}(C_\bullet(A))$$

for any algebra  $A$ , which we shall explore in the next section.

## 5. OPERATIONS ON THE CYCLIC COMPLEX

The Cartan homotopy formula that follows was first observed by Rinehart in [14] in the case where  $D$  is a derivation, and later in full generality by Getzler in [6]. An elegant and conceptual proof of the Cartan homotopy formula can be found in [8]. Our conventions vary slightly, and are more like those in [16].

To simplify the notation of what follows, the elementary tensor  $a_0 \otimes a_1 \otimes \dots \otimes a_n \in C_n(A)$  will be written as  $(a_0, a_1, \dots, a_n)$ . All operators that are defined in this section are given algebraically on elementary tensors, and extend to continuous linear operators on the corresponding projective tensor products.

All commutators of operators that follow are graded commutators. That is, if  $S$  and  $T$  are homogenous operators of degree  $|S|$  and  $|T|$ , then

$$[S, T] = ST - (-1)^{|S||T|}TS.$$

### 5.1. Lie derivatives, contractions, and the Cartan homotopy formula.

Given a Hochschild cochain  $D \in C^k(A, A)$ , the *Lie derivative along  $D$*  is the operator  $L_D \in \text{End}(C_\bullet(A))$  of degree  $-|D| = 1 - k$  given by

$$\begin{aligned} L_D(a_0, \dots, a_n) &= \sum_{i=1}^{n-|D|} (-1)^{|D|(i-1)} (a_0, \dots, D(a_i, \dots, a_{i+|D|}), \dots, a_n) \\ &\quad + \sum_{j=0}^{|D|} (-1)^{|D|+nj} (D(a_{n-j+1}, \dots, a_n, a_0, \dots, a_{|D|-j}), a_{k-j}, \dots, a_{n-j}). \end{aligned}$$

The second sum is taken over all cyclic permutations of the  $a_i$  such that  $a_0$  is within  $D$ . In the case  $D \in C^1(A, A)$ , the above formula is just

$$L_D(a_0, \dots, a_n) = \sum_{i=0}^n (a_0, \dots, a_{i-1}, D(a_i), a_{i+1}, \dots, a_n).$$

As another example, one can consider the multiplication map  $m : A \hat{\otimes} A \rightarrow A$ . Though  $m \notin C^2(A, A)$ , the formula for  $L_m$  still gives a well-defined operator on  $C_\bullet(A)$ , and  $L_m = -b$ .

**Proposition 5.1.** *If  $D \in C^\bullet(A, A)$ , then  $[b, L_D] = -L_{\delta D}$  and  $[B, L_D] = 0$ .*

Note that Proposition 5.1 implies that the Hochschild cohomology  $H^\bullet(A, A)$  acts via Lie derivatives on both the Hochschild homology  $H_\bullet(A)$  and the periodic cyclic homology  $HP_\bullet(A)$ . These actions are Lie algebra actions, as one can show that

$$[L_D, L_E] = L_{[D, E]}.$$

Given a  $k$ -cochain  $D \in C^k(A, A)$ , the *contraction with  $D$*  is the operator  $\iota_D \in \text{End}(C_\bullet(A))$  of degree  $-k$  given by

$$\iota_D(a_0, \dots, a_n) = (a_0 D(a_1, \dots, a_k), a_{k+1}, \dots, a_n).$$

**Proposition 5.2.** *For any  $D \in C^\bullet(A, A)$ ,  $[b, \iota_D] = \iota_{\delta D}$ .*

Although  $\iota_D$  behaves nicely on the Hochschild complex, it does not interact well with the differential  $B$ , and needs to be adjusted for the cyclic complex. Given  $D \in C^k(A, A)$ , let  $S_D$  denote the operator on  $C_\bullet(A)$  of degree  $2 - k$  given by

$$\begin{aligned} S_D(a_0, \dots, a_n) &= \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} (-1)^{(k-1)(j-1)+(n+k-1)i} \\ &\quad (1, a_{n-i+1}, \dots, a_n, a_0, \dots, a_{j-1}, D(a_j, \dots, a_{j+k-1}), a_{j+k}, \dots, a_{n-i}). \end{aligned}$$

The sum is over all cyclic permutations with  $D$  appearing to the right of  $a_0$ . Given  $D \in C^\bullet(A, A)$ , the *cyclic contraction with  $D$*  is the operator

$$I_D = \iota_D + S_D.$$

**Theorem 5.3** (Cartan homotopy formula). *For any  $D \in C^\bullet(A, A)$ ,*

$$[b + B, I_D] = L_D + I_{\delta D}.$$

Theorem 5.3 implies that the Lie derivative along a Hochschild cocycle  $D \in C^\bullet(A, A)$  is continuously chain homotopic to zero in the periodic cyclic complex, and hence this action of  $H^\bullet(A, A)$  on  $HP_\bullet(A)$  is zero.

The results of this section can be summarized in another way. Recall that if  $(C^\bullet, d_C)$  and  $(D^\bullet, d_D)$  are two cochain complexes over  $R$ , then  $\text{Hom}_R(C, D)$  is also a cochain complex. The degree  $n$  part of  $\text{Hom}_R(C, D)$  is the space of all homogeneous  $R$ -linear maps of degree  $n$ , and the coboundary map  $\partial$  is given by

$$\partial F = d_D \circ F - (-1)^{|F|} F \circ d_C$$

on homogeneous elements. Thus,  $F$  is a cocycle if and only if  $F$  is a chain map, and  $G$  is a coboundary if and only if  $G$  is chain homotopic to zero.

The periodic cyclic complex  $C_{\text{per}}(A)$  can be viewed as a  $\mathbb{Z}/2$ -graded cochain complex, and so  $\text{End}_R(C_{\text{per}}(A))$  is a  $\mathbb{Z}/2$ -graded cochain complex whose coboundary map is given by the graded commutator with  $b + B$ . Let

$$\text{Op}(A) = \text{Hom}_R(C^\bullet(A, A), \text{End}_R(C_{\text{per}}(A))),$$

and let  $\partial$  denote the boundary map in  $\text{Op}(A)$ . Given  $\Phi \in \text{Op}(A)$  and  $D \in C^\bullet(A, A)$ , we shall write  $\Phi_D := \Phi(D)$ . So

$$(\partial\Phi)_D = [b + B, \Phi_D] - (-1)^{|\Phi|} \Phi_{\delta D}.$$

Note that the Lie derivative  $L$  and the cyclic contraction  $I$  are elements of  $\text{Op}(A)$  of odd and even degrees respectively. Theorem 5.3 is exactly the statement

$$\partial I = L.$$

So it follows from this that  $\partial L = 0$ , i.e.

$$[b + B, L_D] = -L_{\delta D},$$

which is roughly the content of Proposition 5.1.

**5.2. Some higher operations.** The Lie derivative and contraction operations of the previous section have multiple generalizations, see e.g. [6] or [16]. We shall need just one of these. For  $X, Y \in C^1(A, A)$ , define the operators  $L\{X, Y\}$  and  $I\{X, Y\}$  on  $C_\bullet(A)$  by

$$\begin{aligned} L\{X, Y\}(a_0, \dots, a_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n (a_0, \dots, X(a_i), \dots, Y(a_j), \dots, a_n) \\ &\quad + \sum_{i=1}^n (Y(a_0), a_1, \dots, X(a_i), \dots, a_n). \end{aligned}$$

and

$$\begin{aligned} I\{X, Y\}(a_0, \dots, a_n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sum_{m=0}^{n-j} (-1)^{nm} \\ &\quad (1, a_{n-m+1}, \dots, a_n, a_0, \dots, X(a_i), \dots, Y(a_j), \dots, a_{n-m}). \end{aligned}$$

The following formula appears in [6], with slightly different conventions.

**Theorem 5.4.** *If  $X$  and  $Y$  are derivations, then*

$$[b + B, I\{X, Y\}] = L\{X, Y\} + I_{X \smile Y} - I_Y I_X.$$

**Corollary 5.5.** *If  $X$  and  $Y$  are derivations, then*

$$[b + B, L\{X, Y\}] = -L_{X \cup Y} + L_Y I_X - I_Y L_X.$$

*Proof.* The formula follows immediately by applying the commutator with  $b + B$  to the formula of Theorem 5.4 and using Theorem 5.3.  $\square$

## 6. CONNECTIONS AND PARALLEL TRANSPORT

Since we are only dealing with one-parameter deformations, we shall only treat connections on  $C^\infty(J)$ -modules where the interval  $J$  represents the parameter space. As there is only one direction to differentiate in, a connection is determined by its covariant derivative. In what follows, we shall identify the two notions, and will commonly refer to covariant differential operators as connections.

Let  $M$  be a locally convex  $C^\infty(J)$ -module. A *connection* on  $M$  is a continuous  $\mathbb{C}$ -linear map  $\nabla : M \rightarrow M$  such that

$$\nabla(f \cdot m) = f \cdot \nabla m + f' \cdot m \quad \forall f \in C^\infty(J), m \in M.$$

It is immediate from this Leibniz rule that the difference of two connections is a continuous  $C^\infty(J)$ -linear map. Further, given any connection  $\nabla$  and continuous  $C^\infty(J)$ -linear map  $F : M \rightarrow M$ , the operator  $\nabla + F$  is also a connection. So if the space of connections is nonempty, then it is an affine space parametrized by the space  $\text{End}_{C^\infty(J)}(M)$  of continuous  $C^\infty(J)$ -linear endomorphisms.

**Proposition 6.1.** *If  $\nabla$  is a connection on  $C^\infty(J, X)$ , where  $X$  is a locally convex vector space, then*

$$\nabla = \frac{d}{dt} + F,$$

where  $F : C^\infty(J, X) \rightarrow C^\infty(J, X)$  is some continuous  $C^\infty(J)$ -linear map.

An element in the kernel of a connection  $\nabla$  will be called a *parallel section* for  $\nabla$ . Suppose  $M$  and  $N$  are two locally convex  $C^\infty(J)$ -modules with connections  $\nabla_M$  and  $\nabla_N$  respectively. We shall say that a continuous  $C^\infty(J)$ -linear map  $F : M \rightarrow N$  is *parallel* if  $F \circ \nabla_M = \nabla_N \circ F$ , that is if the diagram

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \nabla_M \downarrow & & \downarrow \nabla_N \\ \check{M} & \xrightarrow{F} & \check{N} \end{array}$$

commutes. Thus a parallel map sends parallel sections to parallel sections.

**Proposition 6.2.** *Given locally convex  $C^\infty(J)$ -modules  $M$  and  $N$  with connections  $\nabla_M$  and  $\nabla_N$ ,*

- (i) *the operator  $\nabla_M \otimes 1 + 1 \otimes \nabla_N$  is a connection on  $M \hat{\otimes}_{C^\infty(J)} N$ .*
- (ii) *the operator  $\nabla_{M^*}$  on  $M^* = \text{Hom}_{C^\infty(J)}(M, C^\infty(J))$  given by*

$$(\nabla_{M^*} \varphi)(m) = \frac{d}{dt} \varphi(m) - \varphi(\nabla_M m)$$

*is a connection.*

The definition of  $\nabla_{M^*}$  ensures that the canonical pairing

$$\langle \cdot, \cdot \rangle : M^* \otimes_{C^\infty(J)} M \rightarrow C^\infty(J)$$

is a parallel map, where we consider  $C^\infty(J)$  with the connection  $\frac{d}{dt}$ . This is just another way of saying that

$$\frac{d}{dt} \langle \varphi, m \rangle = \langle \nabla_{M^*} \varphi, m \rangle + \langle \varphi, \nabla_M m \rangle.$$

**6.1. Parallel transport in free modules.** We will be interested in identifying when we can perform parallel transport along a connection. Suppose  $M$  is a locally convex  $C^\infty(J)$ -module of the form

$$M = C^\infty(J, X)$$

for some complete locally convex vector space  $X$ . We will think of  $M$  as sections of bundle whose fiber over  $t \in J$  is  $M_t \cong X$ . Although all the fibers are the same topological vector space, we introduce the notation  $M_t$  because we will eventually consider examples in which each  $M_t$  will contain additional structure that will depend on  $t$ .

**Definition 6.3.** We shall say that a connection  $\nabla$  on  $M$  is *integrable* if for every  $t \in J$  and every  $m_t \in M_t$ , there exists a unique parallel section  $m \in M$  through  $m_t$ , and moreover, the association  $m_t \mapsto m$  is a continuous map from  $M_t$  to  $M$ .

In other words,  $\nabla$  is integrable if there is a unique  $m \in M$  that satisfies the differential equation

$$\nabla m = 0, \quad m(t) = m_t,$$

and the solution depends continuously on the initial condition. In this case, the parallel transport operator

$$P^\nabla : M_t \rightarrow M_s$$

between any two fibers is defined by

$$P^\nabla(m_t) = m(s)$$

where  $m$  is the unique parallel sections through  $m_t$ . The map  $P^\nabla : M_t \rightarrow M_s$  is an isomorphism of topological vector spaces for all  $t, s \in J$ , as its inverse is  $P^\nabla : M_s \rightarrow M_t$ . Notice that the connection  $\frac{d}{dt}$  in the free module  $C^\infty(J, X)$  is always integrable.

Now suppose  $X$  and  $Y$  are locally convex spaces and  $M = C^\infty(J, X)$  and  $N = C^\infty(J, Y)$ . Suppose  $F : M \rightarrow N$  is a continuous  $C^\infty(J)$ -linear map and  $\{F_t : M_t \rightarrow N_t\}_{t \in J}$  is the smooth family of continuous linear maps associated to  $F$  as in Proposition 2.4.

**Proposition 6.4.** *In the above situation, if  $F : M \rightarrow N$  is parallel with respect to integrable connections on  $M$  and  $N$ , then the diagram*

$$\begin{array}{ccc} M_t & \xrightarrow{F_t} & N_t \\ P^{\nabla_M} \downarrow & & \downarrow P^{\nabla_N} \\ \check{M}_s & \xrightarrow{F_s} & \check{N}_s \end{array}$$

*commutes for all  $t, s \in J$ .*



*Proof.* Given  $m_t \in M_t$ , let  $m \in M$  be the unique parallel section through  $m_t$ . Then  $F(m)$  is the unique parallel section through  $F(m)(t) = F_t(m_t)$ . Consequently,

$$P^{\nabla_N}(F_t(m_t)) = F(m)(s) = F_s(m(s)) = F_s(P^{\nabla_M}(m_t)).$$

□

6.1.1. *The Banach case.* We now consider the special case of  $M = C^\infty(J, X)$  where  $X$  is a Banach space. As we shall see, any connection  $\nabla$  on  $M$  is integrable. Recall by Proposition 6.1 that  $\nabla = \frac{d}{dt} - F$  for some continuous  $C^\infty(J)$ -linear map  $F : C^\infty(J, X) \rightarrow C^\infty(J, X)$ . By Proposition 2.4,  $F$  is given by a smooth family  $\{F_t\}_{t \in J}$  of continuous linear maps on  $X$ . Thus, the initial value problem we wish to solve is

$$x'(t) = F_t(x), \quad x(t_0) = x_0.$$

The following well-known theorem from differential equations says that there are always unique solutions.

**Theorem 6.5.** (*Existence and uniqueness for linear ODE's*) *If  $X$  is a Banach space and  $\{F_t\}_{t \in J}$  is a smooth family of continuous linear maps on  $X$ , then there is a unique global solution  $x \in C^\infty(J, X)$  to the initial value problem*

$$x'(t) = F_t(x(t)), \quad x(t_0) = x_0,$$

given by

$$x(t) = x_0 + \sum_{n=1}^{\infty} \int_{t_0}^t \int_{t_0}^{s_1} \dots \int_{t_0}^{s_{n-1}} (F_{s_1} \circ \dots \circ F_{s_n})(x_0) ds_n \dots ds_1.$$

Moreover, using the explicit form of the solution  $x$ , one can show that the map  $X \rightarrow C^\infty(J, X)$  given by  $x_0 \mapsto x$  is continuous. Thus,  $\nabla$  is integrable.

**Corollary 6.6.** *If  $X$  is a Banach space and  $\nabla$  is a connection on  $M = C^\infty(J, X)$ , then the parallel transport map*

$$P^\nabla : M_t \rightarrow M_s$$

*exists for all  $t, s \in J$  and is an isomorphism of topological vector spaces.*

A particular notable special case is when the differential equation has “constant coefficients,” that is,  $F_t$  is independent of  $t$ . The initial value problem is then

$$x'(t) = F_0(x(t)), \quad x(t_0) = x_0$$

for some continuous linear map  $F_0 : X \rightarrow X$ . In this case, the explicit solution of Theorem 6.5 becomes

$$x(t) = \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{n!} F_0^n(x_0) = \exp((t - t_0)F_0)(x_0),$$

where the exponential is taken in the Banach algebra of bounded linear operators on  $X$ .

**Remark 6.7.** Once we start considering other classes of locally convex vector spaces, the above existence and uniqueness theorem for linear ODE's becomes false. The issue is that the infinite sum in the formula for the solution need not converge. Indeed, even in the case where  $X$  is a nuclear Fréchet space and  $F_0 : X \rightarrow X$  is a continuous linear map, the infinite series  $\exp(F_0)$  need not converge in  $\text{Hom}(X, X)$ . Thus, it is not true in the Fréchet case that every connection is integrable.

**6.2. Nilpotent perturbations of integrable connections.** Suppose  $\nabla_0$  is an integrable connection on a locally convex  $C^\infty(J)$ -module  $M$  and  $\nabla = \nabla_0 - F$  is another connection. It is useful to identify conditions on  $F$  which ensure that  $\nabla$  is integrable. This happens, for example, when  $F$  is nilpotent, i.e. there is a positive integer  $k$  for which  $F^k = 0$ . In this case, the exponential

$$\exp(F) = \sum_{n=0}^{k-1} \frac{F^n}{n!}$$

exists, as it is a finite algebraic sum.

In what follows, the assumption  $[\nabla_0, F] = 0$  can be thought of as a “constant coefficient” condition. The relationship to constant coefficient differential equations is that if the map  $F$  comes from the smooth family  $\{F_t\}_{t \in J}$  of continuous linear maps, then  $[\frac{d}{dt}, F] = 0$  if and only if  $F_t$  is independent of  $t$ .

**Proposition 6.8.** *Suppose  $\nabla = \nabla_0 - F$  is a connection on  $M$  where  $\nabla_0$  is an integrable connection and  $F$  is nilpotent such that  $[\nabla_0, F] = 0$ . Then  $\nabla$  is integrable.*

*Proof.* If  $m \in M$  is a  $\nabla_0$ -parallel section through  $m_0 \in M_{t_0}$ , then it is straightforward to verify that

$$\exp((t - t_0)F)(m) = \sum_{n=0}^{k-1} \frac{(t - t_0)^n}{n!} F^n(m)$$

is a  $\nabla$ -parallel section through  $m_0$ . Likewise, if  $\tilde{m}$  is a  $\nabla$ -parallel section through  $m_0$ , then  $\exp(-(t - t_0)F)(\tilde{m})$  is a  $\nabla_0$ -parallel section through  $m_0$ . Since,

$$\exp((t - t_0)F) \exp(-(t - t_0)F) = \exp(-(t - t_0)F) \exp((t - t_0)F) = 1,$$

this establishes a one-to-one correspondence between  $\nabla_0$ -parallel sections through  $m_0$  and  $\nabla$ -parallel sections through  $m_0$ . Therefore,  $\nabla_0$  is integrable if and only if  $\nabla$  is integrable. The required continuity condition holds because  $\exp((t - t_0)F)$  is a continuous operator on  $M$ .  $\square$

**6.3. Bundle of algebras.** Let  $A$  be the section algebra of a unit-preserving smooth one-parameter deformation. We shall only consider connections on  $A$  for which  $\nabla(1) = 0$ . Given such a connection  $\nabla$ , let  $E = -\delta\nabla$ , so that

$$\nabla(a_1 a_2) = \nabla(a_1) a_2 + a_1 \nabla(a_2) + E(a_1, a_2).$$

From its definition, it is clear that  $\delta E = 0$ , that is,  $E$  is a Hochschild 2-cocycle. However, from the Leibniz rule for  $\nabla$ , one can check that  $E$  is  $C^\infty(J)$ -bilinear. So  $E$  defines a cohomology class in  $H_{C^\infty(J)}^2(A, A)$ . It may appear from its definition that  $E$  is a coboundary, but this is not necessarily true because  $\nabla$  is not a  $C^\infty(J)$ -linear map.

**Proposition 6.9.** *The cohomology class of  $E$  in  $H_{C^\infty(J)}^2(A, A)$  is independent of the choice of connection. Moreover,  $[E] = 0$  if and only if  $A$  possesses a connection that is a derivation.*

*Proof.* If  $\nabla$  and  $\nabla'$  are two connections with corresponding cocycles  $E$  and  $E'$ , then  $\nabla' = \nabla + F$  for some  $F \in C_{C^\infty(J)}^1(A, A)$ . So

$$E' = -\delta(\nabla + F) = E - \delta F,$$

which shows that  $[E] = [E']$ .

If  $\nabla$  is a connection that is a derivation, then  $E = -\delta\nabla = 0$ . Conversely, if  $\nabla$  is any connection on  $A$  and  $[E] = 0$ , then  $-\delta\nabla = \delta F$  for some  $F \in C^1_{C^\infty(J)}(A, A)$ . That is,  $\delta(\nabla + F) = 0$ , and so  $\nabla + F$  is a connection that is a derivation.  $\square$

**Proposition 6.10.** *If  $\nabla$  is an integrable connection on  $A$  which is also a derivation, then the parallel transport map  $P^\nabla : A_t \rightarrow A_s$  is an algebra isomorphism for all  $t, s \in J$ .*

*Proof.* If  $\nabla$  is integrable, it is not difficult to check that the connection  $\nabla \otimes 1 + 1 \otimes \nabla$  on  $A \otimes_{C^\infty(J)} A$  is integrable with parallel transport maps of the form  $P^\nabla \otimes P^\nabla$ . Then  $\nabla$  is a derivation if and only if the multiplication  $m : A \otimes_{C^\infty(J)} A \rightarrow A$  is a parallel map. Thus, the result follows from Proposition 6.4.  $\square$

From this, we see that the cohomology class of  $E$  provides an obstruction to the existence of algebra isomorphisms between the fibers provided by parallel transport.

**6.4. Bundle of complexes.** Here, we consider the situation where  $M = C^\infty(J, X)$  is the space of sections of a smooth fields of complexes. That is, suppose  $X$  is a graded locally convex space and  $\partial : M \rightarrow M$  is a continuous  $C^\infty(J)$ -linear map of degree  $-1$  for which  $\partial^2 = 0$ . By Proposition 2.4, there is a smooth family  $\{\partial_t\}_{t \in J}$  of continuous linear maps on  $X$  that satisfy  $\partial_t^2 = 0$ . So each fiber  $M_t := (X, \partial_t)$  is a complex. The main example for us will be when  $M = C_{\text{per}}(A)$  for the section algebra  $A$  of a smooth one-parameter deformation

We shall be interested in connections for which the boundary map is parallel. Following the algebra case, let  $G = [\partial, \nabla] : M \rightarrow M$ . Recall that  $\text{End}_{C^\infty(J)}(M)$  is a chain complex whose boundary map is given by the commutator with  $\partial$ . By definition, it is clear that  $[\partial, G] = 0$ . If  $\nabla_n$  denotes the degree  $n$  component of  $\nabla$ , then  $\nabla_0$  is a connection and  $\nabla_n$  is  $C^\infty(J)$ -linear for  $n \neq 0$ . Thus, we see that  $G_{n-1} = [\partial, \nabla_n]$  is  $C^\infty(J)$ -linear for each  $n$ , so  $G$  defines a homology class in  $H_\bullet(\text{End}_{C^\infty(J)}(M))$ , and  $[G] = [G_{-1}]$ .

**Proposition 6.11.** *The homology class  $[G_{-1}] \in H_\bullet(\text{End}_{C^\infty(J)}(M))$  is independent of the choice of connection  $\nabla$ . Moreover,  $[G_{-1}] = 0$  if and only if  $M$  possesses a connection that is a chain map.*

*Proof.* The proof is completely analogous to that of Proposition 6.9.  $\square$

**Proposition 6.12.** *If  $\nabla$  is a connection on  $M$  that is also a chain map, then  $\nabla$  descends to a connection  $\nabla_*$  on the  $C^\infty(J)$ -module  $H_\bullet(M)$ . Moreover, if  $\nabla$  is integrable, then  $P^\nabla : M_t \rightarrow M_s$  is an isomorphism of complexes for all  $t, s \in J$ , and thus descends to an isomorphism  $P_*^\nabla : H_\bullet(M_t) \rightarrow H_\bullet(M_s)$ .*

*Proof.* The first statement follows from standard elementary homological algebra. Now  $\nabla$  is a chain map if and only if  $\partial$  is parallel with respect to  $\nabla$ . So the fact that  $P^\nabla : M_t \rightarrow M_s$  is a chain map follows from Proposition 6.4.  $\square$

The previous proposition had the rather strong hypothesis that the connection  $\nabla$  is integrable on  $M$ . Consequently, the conclusion that the complexes are fiberwise isomorphic is too much to hope for in many situations. Even if  $\nabla$  is not integrable on  $M$ , one still has the possibility that  $\nabla_*$  is integrable on  $H_\bullet(M)$ .

## 7. THE GAUSS-MANIN CONNECTION

**7.1. Homological version.** Let  $A$  denote the section algebra of a smooth one-parameter deformation of locally convex algebras  $\{A_t\}_{t \in J}$ . Unless specified otherwise, all chain groups and homology groups associated to  $A$  that follow are over the ground ring  $C^\infty(J)$ .

Our goal is to construct a connection on  $C_{\text{per}}(A)$  that commutes with  $b + B$ . Let  $\nabla$  be a connection on  $A$  and let  $E = -\delta\nabla$  as in §6.3. Using Proposition 6.2,  $\nabla$  extends<sup>3</sup> to a connection on  $C_{\text{per}}(A)$ , which is given by  $L_\nabla$ . By Proposition 6.11,  $C_{\text{per}}(A)$  possesses a connection that is a chain map if and only if

$$G := [b + B, L_\nabla] = L_E$$

is a boundary in the complex  $\text{End}_{C^\infty(J)}(C_{\text{per}}(A))$ . But Theorem 5.3 says that

$$[b + B, I_E] = L_E.$$

As  $I_E$  is a continuous  $C^\infty(J)$ -linear map, the *Gauss-Manin connection*

$$\nabla_{GM} = L_\nabla - I_E$$

is a connection on  $C_{\text{per}}(A)$  and a chain map.

**Proposition 7.1.** *The Gauss-Manin connection  $\nabla_{GM}$  commutes with the differential  $b + B$  and hence induces a connection on the  $C^\infty(J)$ -module  $HP_\bullet(A)$ . Moreover, the induced connection is independent of the choice of connection  $\nabla$  on  $A$ .*

*Proof.* We have already established the first claim. For another connection  $\nabla'$ , let

$$\nabla'_{GM} = L_{\nabla'} - I_{E'}$$

be the corresponding Gauss-Manin connection. Then

$$\nabla' = \nabla + F, \quad E' = E - \delta F$$

for some  $C^\infty(J)$ -linear map  $F : A \rightarrow A$ . Thus,

$$\nabla'_{GM} - \nabla_{GM} = L_F + I_{\delta F} = [b + B, I_F],$$

by Theorem 5.3. We conclude that the Gauss-Manin connection is unique up to continuous chain homotopy.  $\square$

**Corollary 7.2.** *If  $A$  admits a connection  $\nabla$  which is also a derivation, then the induced map on  $HP_\bullet(A)$  is given by*

$$\nabla_{GM}[\omega] = [L_\nabla\omega].$$

Note that if an integrable connection  $\nabla$  on  $A$  is also a derivation, then the parallel transport map  $P^\nabla : A_t \rightarrow A_s$  is an algebra isomorphism by Proposition 6.10. In this case, one can show that  $\nabla_{GM}$  is integrable on  $C_{\text{per}}(A)$ , and the parallel transport isomorphism

$$P^{\nabla_{GM}} : HP_\bullet(A_t) \rightarrow HP_\bullet(A_s)$$

is the same as the map on periodic cyclic homology induced by the algebra isomorphism  $P^\nabla : A_t \rightarrow A_s$ .

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<sup>3</sup>Since we are working on the normalized complex, it is important that we assume  $\nabla(1) = 0$  here.

**Proposition 7.3.** *If  $\omega \in C_{\text{per}}^{C^\infty(J)}(A)$  is a cycle that lifts to a cycle  $\tilde{\omega} \in C_{\text{per}}^{\mathbb{C}}(A)$ , then*

$$\nabla_{GM}[\omega] = 0$$

*in  $HP_{\bullet}^{C^\infty(J)}(A)$ .*

*Proof.* Let  $\nabla_{GM}^{\mathbb{C}} = L_{\nabla} - I_E$ , viewed as an operator on  $C_{\text{per}}^{\mathbb{C}}(A)$ . By Theorem 5.3,

$$\nabla_{GM}^{\mathbb{C}} = L_{\nabla} + I_{\delta\nabla} = [b + B, I_{\nabla}]$$

and so  $\nabla_{GM}^{\mathbb{C}}$  is the zero operator on  $HP_{\bullet}^{\mathbb{C}}(A)$ . Thus, at the level of homology, we have

$$\nabla_{GM} \circ \pi = \pi \circ \nabla_{GM}^{\mathbb{C}} = 0$$

where  $\pi : HP_{\bullet}^{\mathbb{C}}(A) \rightarrow HP_{\bullet}^{C^\infty(J)}(A)$  is the map induced by the quotient map. By hypothesis,  $\omega$  is in the image of  $\pi$ .  $\square$

Note that the homotopy used in the previous proof does not imply that  $\nabla_{GM}$  is zero on  $HP_{\bullet}^{C^\infty(J)}(A)$ . The reason is that the operator  $I_{\nabla}$  is not a well-defined operator on the quotient complex  $C_{\text{per}}^{C^\infty(J)}(A)$ .

**Corollary 7.4.** *If  $P \in M_N(A)$  is an idempotent, then*

$$\nabla_{GM}[\text{ch } P] = 0.$$

*in  $HP_0(A)$ .*

**Corollary 7.5.** *If  $U \in M_N(A)$  is an invertible, then*

$$\nabla_{GM}[\text{ch } U] = 0.$$

*in  $HP_1(A)$ .*

The proofs are immediate from the previous proposition in the case  $N = 1$ . If  $\nabla$  is a connection on  $A$ , then  $\nabla_N = 1 \otimes \nabla$  is a connection on  $M_N(A) \cong M_N(\mathbb{C}) \otimes A$ . Moreover,  $E_N = -\delta\nabla_N$  is given by

$$E_N(u_1 \otimes a_1, u_2 \otimes a_2) = u_1 u_2 \otimes E(a_1, a_2).$$

One can show that

$$L_{\nabla} \circ T = T \circ L_{\nabla_N}, \quad I_E \circ T = T \circ I_{E_N},$$

where  $T : C_{\text{per}}(M_N(A)) \rightarrow C_{\text{per}}(A)$  is the generalized trace. Thus,  $T$  is parallel with respect to the Gauss-Manin connections, and the corollaries follow because  $T$  maps parallel sections to parallel sections.

**7.2. Cohomological version.** We define  $\nabla^{GM}$  on  $C^{\text{per}}(A)$  to be the dual connection of  $\nabla_{GM}$  as in Proposition 6.2. In terms of the canonical pairing,

$$\langle \nabla^{GM} \varphi, \omega \rangle = \frac{d}{dt} \langle \varphi, \omega \rangle - \langle \varphi, \nabla_{GM} \omega \rangle.$$

It is straightforward to verify that  $\nabla^{GM}$  commutes with  $b+B$  and therefore induces a connection on  $HP^{\bullet}(A)$ . The connections  $\nabla_{GM}$  and  $\nabla^{GM}$  are compatible in the sense that

$$\frac{d}{dt} \langle [\varphi], [\omega] \rangle = \langle \nabla^{GM} [\varphi], [\omega] \rangle + \langle [\varphi], \nabla_{GM} [\omega] \rangle,$$

for all  $[\varphi] \in HP^{\bullet}(A)$ ,  $[\omega] \in HP_{\bullet}(A)$ .

**7.3. Naturality of  $\nabla_{GM}$ .** The following naturality property of  $\nabla_{GM}$  says that algebra maps between deformations induce parallel maps at the level of periodic cyclic homology.

**Proposition 7.6.** *Let  $A$  and  $B$  denote the section algebras of two smooth one-parameter deformations over the same parameter space  $J$ , and let  $F : A \rightarrow B$  be a continuous  $C^\infty(J)$ -linear unital algebra map. Then  $\nabla_{GM}$  is natural in the sense that the following diagram commutes.*

$$\begin{array}{ccc} HP_\bullet(A) & \xrightarrow{F_*} & HP_\bullet(B) \\ \nabla_{GM}^A \downarrow & & \downarrow \nabla_{GM}^B \\ HP_\bullet(A) & \xrightarrow{F_*} & HP_\bullet(B) \end{array}$$

*Proof.* Let  $\nabla^A$  and  $\nabla^B$  denote connections on  $A$  and  $B$  with respective cocycles  $E^A$  and  $E^B$ , and let  $F_* : C_{\text{per}}(A) \rightarrow C_{\text{per}}(B)$  be the induced map of complexes. For

$$h = F_* I_{\nabla^A} - I_{\nabla^B} F_*,$$

we have

$$\begin{aligned} [b + B, h] &= F_*[b + B, I_{\nabla^A}] - [b + B, I_{\nabla^B}]F_* \\ &= F_*(L_{\nabla^A} - I_{E^A}) - (L_{\nabla^B} - I_{E^B})F_* \\ &= F_*\nabla_{GM}^A - \nabla_{GM}^B F_* \end{aligned}$$

shows that the diagram commutes up to continuous chain homotopy. The problem is that  $I_{\nabla^A}$  and  $I_{\nabla^B}$  are not well-defined operators on the complexes  $C_{\text{per}}^{C^\infty(J)}(A)$  and  $C_{\text{per}}^{C^\infty(J)}(B)$  respectively. However, one can show that because of the Leibniz rule,  $h$  descends to a map of quotient complexes such that the following diagram

$$\begin{array}{ccc} C_{\text{per}}^{\mathbb{C}}(A) & \xrightarrow{h} & C_{\text{per}}^{\mathbb{C}}(B) \\ \pi \downarrow & & \downarrow \pi \\ C_{\text{per}}^{C^\infty(J)}(A) & \xrightarrow{\bar{h}} & C_{\text{per}}^{C^\infty(J)}(B) \end{array}$$

commutes, and consequently  $[b + B, \bar{h}] = F_*\nabla_{GM}^A - \nabla_{GM}^B F_*$  as desired.  $\square$

As a simple application of Proposition 7.6, we get a proof of the homotopy invariance property of periodic cyclic homology by considering morphisms between trivial deformations.

**Corollary 7.7.** (*Homotopy Invariance*) *Let  $A_0$  and  $B_0$  be complex Fréchet algebras and let  $\{F_t : A_0 \rightarrow B_0\}_{t \in J}$  be a smooth family of unital algebra maps. Then the induced map*

$$(F_t)_* : HP_\bullet(A_0) \rightarrow HP_\bullet(B_0)$$

*is independent of  $t$ .*

*Proof.* Let  $A = C^\infty(J, A_0)$  and  $B = C^\infty(J, B_0)$  be the algebras corresponding to the trivial deformations of  $A_0$  and  $B_0$  respectively over  $J$ . Then  $\{F_t\}_{t \in J}$  is a

morphism between these trivial deformations, and by Proposition 3.5 gives rise to a continuous  $C^\infty(J)$ -linear algebra map  $F : A \rightarrow B$  such that

$$F(a)(t) = F_t(a(t)).$$

Using the canonical connections given by  $\frac{d}{dt}$  on both  $A$  and  $B$ , we see that  $\nabla_{GM}^A$  and  $\nabla_{GM}^B$  are given by  $\frac{d}{dt}$  under the identification  $C_{\text{per}}(A) \cong C^\infty(J, C_{\text{per}}^\mathbb{C}(A_0))$  and similarly for  $B$ . Given a cycle  $\omega \in C_{\text{per}}(A_0)$ , it can also be viewed as a “constant” cycle in  $C_{\text{per}}(A)$ , and then Proposition 7.6 implies that

$$\frac{d}{dt}[F_t\omega] = F_t \frac{d}{dt}[\omega] = 0.$$

So there is  $\eta \in C^\infty(J, C_{\text{per}}^\mathbb{C}(B_0))$  such that

$$\frac{d}{dt}F_t(\omega) = (b + B)(\eta(t)).$$

But, by the fundamental theorem of calculus,

$$F_{t_1}(\omega) - F_{t_0}(\omega) = \int_{t_0}^{t_1} (b + B)(\eta(s))ds = (b + B) \left( \int_{t_0}^{t_1} \eta(s)ds \right)$$

for any  $t_0, t_1 \in J$ . □

Proposition 7.6 can also be used to give an alternative proof that

$$\nabla_{GM}[\text{ch } P] = 0$$

when  $P \in A$  is an idempotent. Indeed, an idempotent in  $A$  is given by a collection of algebra maps

$$\{F_t : \mathbb{C} \rightarrow A_t\}_{t \in J}.$$

Smoothness of  $P$  is equivalent to  $\{F_t\}_{t \in J}$  being a morphism from the trivial deformation over  $J$  with fiber  $\mathbb{C}$  to  $\{A_t\}_{t \in J}$ . Thus by Proposition 3.5, it induces an  $C^\infty(J)$ -linear algebra map

$$F : C^\infty(J) \rightarrow A$$

which maps 1 to  $P$ . Applying Proposition 7.6, we see

$$\nabla_{GM}[\text{ch } P] = \nabla_{GM}F[\text{ch } 1] = F \frac{d}{dt}[\text{ch } 1] = 0.$$

This argument is actually not quite correct, as we require our algebra maps to be unital so that the cyclic complex is functorial. To fix this, we just replace the map  $F_t : \mathbb{C} \rightarrow A_t$  with a map from the unitization  $\tilde{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}$  to  $A_t$  which sends  $1 \in \mathbb{C}$  to  $P_t$  and the adjoined unit to 1.

**7.4. Integrating  $\nabla_{GM}$ .** The very fact that  $\nabla_{GM}$  exists for all smooth one-parameter deformations implies that the problem of proving  $\nabla_{GM}$  is integrable cannot be attacked with methods that are too general. Indeed, one cannot expect periodic cyclic homology to be rigid for all deformations, there are plenty of finite dimensional examples to show this.

**Example 7.8.** For  $t \in \mathbb{R}$ , let  $A_t$  be the two-dimensional algebra generated by an element  $x$  and the unit 1 subject to the relation  $x^2 = t \cdot 1$ . Then  $A_t \cong \mathbb{C} \oplus \mathbb{C}$  as

an algebra when  $t \neq 0$ , and  $A_0$  is the exterior algebra on a one dimensional vector space. Consequently,

$$HP_0(A_t) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}, & t \neq 0 \\ \mathbb{C}, & t = 0. \end{cases}$$

From the point of view of differential equations, one issue is that the periodic cyclic complex is never a Banach space. Even in the case where  $A$  is a Banach algebra (e.g. finite dimensional,) the chain groups  $C_n(A)$  are also Banach spaces, but the periodic cyclic complex

$$C_{\text{per}}(A) = \prod_{n=0}^{\infty} C_n(A)$$

is a Fréchet space, as it is a countable product of Banach spaces. To make matters worse,  $\nabla_{GM}$  contains a degree  $-2$  component in the term  $\iota_E$ . Thus, one cannot reduce the problem to the individual Banach space factors, as the differential equations are hopelessly coupled together. The situation seems worse for the periodic cyclic cochain complex  $C^{\text{per}}(A) \cong C_{\text{per}}(A)^*$ , which is not even metrizable.

However, these are obstructions to proving  $\nabla_{GM}$  is integrable on  $C_{\text{per}}(A)$ , as opposed to  $HP_{\bullet}(A)$ . Our general approach will be to find a new complex that computes  $HP_{\bullet}(A)$  and find a new connection that commutes with the boundary.

## 8. THE $\mathfrak{g}$ -INVARIANT COMPLEX

**8.1.  $\mathfrak{g}$ -invariant chains and cochains.** Suppose that  $\mathfrak{g} \subset C^1(A, A)$  is a Lie subalgebra of derivations on an algebra  $A$ . Then  $\mathfrak{g}$  also acts on  $C_{\bullet}(A)$  by Lie derivatives. Define the  $\mathfrak{g}$ -invariant Hochschild chain group  $C_{\bullet}^{\mathfrak{g}}(A)$  to be

$$C_{\bullet}^{\mathfrak{g}}(A) = C_{\bullet}(A) / \overline{\mathfrak{g} \cdot C_{\bullet}(A)},$$

where  $\overline{\mathfrak{g} \cdot C_{\bullet}(A)}$  is the closed submodule generated by elements of the form  $L_X \omega$  for  $X \in \mathfrak{g}$  and  $\omega \in C_{\bullet}(A)$ . By Proposition 5.1, the operators  $b$  and  $B$  descend to operators on  $C_{\bullet}^{\mathfrak{g}}(A)$ . One can define the  $\mathfrak{g}$ -invariant periodic cyclic complex  $C_{\text{per}}^{\mathfrak{g}}(A)$  accordingly, and its homology is the  $\mathfrak{g}$ -invariant periodic cyclic homology  $HP_{\bullet}^{\mathfrak{g}}(A)$ .

Let  $C_{\mathfrak{g}}^{\bullet}(A, A)$  denote the space of all Hochschild cochains  $D$  for which  $[X, D] = 0$  for all  $X \in \mathfrak{g}$ . If  $D \in C_{\mathfrak{g}}^{\bullet}(A, A)$ , then the formula

$$0 = \delta[X, D] = [\delta X, D] + [X, \delta D]$$

shows that  $C_{\mathfrak{g}}^{\bullet}(A, A)$  is a subcomplex because  $\delta X = 0$ . Its cohomology is the  $\mathfrak{g}$ -invariant Hochschild cohomology  $H_{\mathfrak{g}}^{\bullet}(A, A)$ .

**Proposition 8.1.** *For any  $D \in C_{\mathfrak{g}}^{\bullet}(A)$ , the operators  $L_D$  and  $I_D$  are well-defined on the  $\mathfrak{g}$ -invariant complex  $C_{\bullet}^{\mathfrak{g}}(A)$ .*

*Proof.* If  $X \in C^1(A, A)$  is a derivation and  $D \in C^{\bullet}(A, A)$ , then one can verify directly that

$$[L_X, I_D] = I_{[X, D]}, \quad [L_X, L_D] = L_{[X, D]},$$

from which the proposition follows.  $\square$

**Proposition 8.2.** *If  $X, Y \in C_{\mathfrak{g}}^1(A, A)$ , then  $L\{X, Y\}$  and  $I\{X, Y\}$  are well-defined operators on  $C_{\bullet}^{\mathfrak{g}}(A)$ .*



*Proof.* For any  $X, Y, Z \in C^1(A, A)$ , one can verify directly the identities

$$[L_Z, I\{X, Y\}] = I\{[Z, X], Y\} + I\{X, [Z, Y]\}$$

and

$$[L_Z, L\{X, Y\}] = L\{[Z, X], Y\} + L\{X, [Z, Y]\},$$

and the result follows.  $\square$

**8.2. The abelian case.** Consider the special case of an abelian Lie algebra  $\mathfrak{g}$  of derivations on  $A$ , so that  $\mathfrak{g} \subset C_{\mathfrak{g}}^{\bullet}(A, A)$ . By definition of the invariant complex, the operator  $L_X$  vanishes on  $C_{\bullet}^{\mathfrak{g}}(A)$  for any  $X \in \mathfrak{g}$ .

One of the benefits of working in the  $\mathfrak{g}$ -invariant complex is that the cyclic contraction  $I_X$  is now a chain map when  $X \in \mathfrak{g}$ . Indeed,

$$[b + B, I_X] = L_X = 0$$

in  $C_{\bullet}^{\mathfrak{g}}(A)$ . These contraction operators obey the following algebra as operators on homology.

**Theorem 8.3.** *There is an algebra map  $\chi : \Lambda^{\bullet} \mathfrak{g} \rightarrow \text{End}(HP_{\bullet}^{\mathfrak{g}}(A))$  given by*

$$\chi(X_1 \wedge X_2 \wedge \dots \wedge X_k) = I_{X_1} I_{X_2} \dots I_{X_k}.$$

*Proof.* First observe that  $X \mapsto I_X$  is a linear mapping. Next, we shall show that  $I_X I_X$  is chain homotopic to zero. Observe that

$$0 = L_X L_X = L_{X^2} + 2L\{X, X\}$$

where  $X^2$  denotes the composition of  $X$  with itself. Thus,

$$L\{X, X\} = -L_{\frac{1}{2}X^2}.$$

Next, notice that

$$\delta\left(\frac{1}{2}X^2\right) = -X \smile X.$$

By Theorems 5.4 and 5.3,

$$\begin{aligned} -[b + B, I\{X, X\}] &= -L\{X, X\} - I_{X \smile X} + I_X I_X \\ &= L_{\frac{1}{2}X^2} + I_{\delta(\frac{1}{2}X^2)} + I_X I_X \\ &= [b + B, I_{\frac{1}{2}X^2}] + I_X I_X, \end{aligned}$$

proving that  $I_X I_X$  is continuously chain homotopic to zero. By the universal property of the exterior algebra, the map  $\chi$  exists as asserted.  $\square$

There are some additional simplifications regarding the operator  $L\{X, Y\}$  once we pass to  $C_{\bullet}^{\mathfrak{g}}(A)$ .

**Proposition 8.4.** *For  $X, Y \in \mathfrak{g}$ , the operator  $L\{X, Y\}$  satisfies*

$$L\{X, Y\}(a_0, \dots, a_n) = \sum_{i=0}^{n-1} \sum_{j=i+1}^n (a_0, \dots, X(a_i), \dots, Y(a_j), \dots, a_n)$$

on  $C_{\bullet}^{\mathfrak{g}}(A)$ . Additionally,

$$[b + B, L\{X, Y\}] = -L_{X \smile Y}$$

on  $C_{\bullet}^{\mathfrak{g}}(A)$ .

*Proof.* Notice that

$$\begin{aligned} & L_Y(X(a_0), a_1, \dots, a_n) - L_X(Y(a_0), a_1, \dots, a_n) \\ &= \sum_{j=1}^n (X(a_0), \dots, Y(a_j), \dots, a_n) - \sum_{i=1}^n (Y(a_0), \dots, X(a_i), \dots, a_n), \end{aligned}$$

using the fact that  $[X, Y] = 0$ . So

$$\begin{aligned} & L\{X, Y\}(a_0, \dots, a_n) + L_Y(X(a_0), a_1, \dots, a_n) - L_X(Y(a_0), a_1, \dots, a_n) \\ &= \sum_{i=0}^{n-1} \sum_{j=i+1}^n (a_0, \dots, X(a_i), \dots, Y(a_j), \dots, a_n) \end{aligned}$$

gives the desired conclusion.

The formula

$$[b + B, L\{X, Y\}] = -L_{X \smile Y}$$

follows from Corollary 5.5 in light of the fact that  $L_X = L_Y = 0$  on  $C^\bullet_{\mathfrak{g}}(A)$ .  $\square$

**8.3. Connections on the  $\mathfrak{g}$ -invariant complex.** Now, suppose  $A$  is the section algebra of a smooth one-parameter deformation over  $J$ . Additionally, suppose that  $A$  possesses a connection  $\nabla$  such that

$$E := -\delta\nabla = \sum_{i=1}^N X_i \smile Y_i,$$

where  $X_i, Y_i \in C^1_{C^\infty(J)}(A, A)$  are derivations that mutually commute and satisfy

$$[\nabla, X_i] = [\nabla, Y_i] = 0.$$

Let  $\mathfrak{g} = \text{Span}\{X_i, Y_i | 1 \leq i \leq N\}$  denote the (abelian) Lie subalgebra of  $C^1(A, A)$  generated by the  $X_i$  and  $Y_i$ .

**Proposition 8.5.** *In the above situation, the Gauss-Manin Connection*

$$\nabla_{GM} = L_\nabla - I_E$$

*descends to the  $\mathfrak{g}$ -invariant complex  $C^\bullet_{\mathfrak{g}}(A)$  and therefore to a connection on the  $\mathfrak{g}$ -invariant periodic cyclic homology  $HP^\bullet_{\mathfrak{g}}(A)$ .*

*Proof.* If  $X, Y, Z \in C^1(A, A)$  and  $X$  is a derivation, then

$$[X, Y \smile Z] = [X, Y] \smile Z + Y \smile [X, Z].$$

Thus,  $E \in C^2_{\mathfrak{g}}(A, A)$  and the result follows from Proposition 8.1.  $\square$

Our main reason for working with the  $\mathfrak{g}$ -invariant complex is that we can define another connection on  $HP^\bullet_{\mathfrak{g}}(A)$  which is easier to work with than  $\nabla_{GM}$ . Recall that the connection  $L_\nabla$  satisfies

$$[b + B, L_\nabla] = L_E = \sum_{i=1}^N L_{X_i \smile Y_i}.$$

Thus, by Proposition 8.4,

$$\tilde{\nabla} = L_\nabla + \sum_{i=1}^N L\{X_i, Y_i\}$$

is a connection on  $C_{\text{per}}^{\mathfrak{g}}$  that commutes with  $b + B$  and therefore descends to a connection on  $HP_{\bullet}^{\mathfrak{g}}(A)$ . We emphasize that  $\tilde{\nabla}$  does not commute with  $b + B$  on the ordinary periodic cyclic complex  $C_{\text{per}}(A)$ .

**Remark 8.6.** The connection  $\tilde{\nabla}$  is very natural from a certain perspective. Let  $\mathcal{H}$  be the ‘‘Heisenberg-like’’ Hopf algebra whose underlying algebra is the symmetric algebra  $S(\mathfrak{g} \oplus \text{Span}\{\nabla\})$ . The coproduct  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  is the unique algebra map that satisfies

$$\begin{aligned}\Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i, & \Delta(Y_i) &= Y_i \otimes 1 + 1 \otimes Y_i, \\ \Delta(\nabla) &= \nabla \otimes 1 + 1 \otimes \nabla + \sum_{i=1}^N X_i \otimes Y_i.\end{aligned}$$

Then  $\mathcal{H}$  acts (as an algebra) on  $A$  in the obvious way, and this action is a Hopf action in the sense that for all  $h \in \mathcal{H}$ ,

$$h(a_1 a_2) = \sum h_{(1)}(a_1) h_{(2)}(a_2),$$

where  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ . When a Hopf algebra  $\mathcal{H}$  acts (say, on the left) on two spaces  $V$  and  $W$ , there is a canonical action, called the *diagonal action*, of  $\mathcal{H}$  on  $V \otimes W$  given by

$$h(v \otimes w) = \sum h_{(1)}(v) \otimes h_{(2)}(w).$$

The connection  $\tilde{\nabla}$  on  $C_n^{\mathfrak{g}}(A)$  is none other than the diagonal action of  $\nabla$  on  $A^{\widehat{\otimes}(n+1)}$  after passing to the quotient.

**Lemma 8.7.** *On the invariant complex  $C_{\bullet}^{\mathfrak{g}}(A)$ ,  $[\nabla_{GM}, \tilde{\nabla}] = 0$ .*

*Proof.* This is a straightforward, though tedious, computation. □

**Proposition 8.8.** *As operators on  $HP_{\bullet}^{\mathfrak{g}}(A)$ ,*

$$\nabla_{GM} = \tilde{\nabla} + \sum_{i=1}^N \chi(X_i \wedge Y_i).$$

*Proof.* We have

$$\begin{aligned}\nabla_{GM} - \tilde{\nabla} &= -I_E - \sum_{i=1}^N L\{X_i, Y_i\} \\ &= -\sum_{i=1}^N \left( I_{X_i \smile Y_i} + L\{X_i, Y_i\} \right) \\ &= -[b + B, \sum_{i=1}^N I\{X_i, Y_i\}] - \sum_{i=1}^N I_{Y_i} I_{X_i} \\ &= -[b + B, \sum_{i=1}^N I\{X_i, Y_i\}] - \sum_{i=1}^N \chi(Y_i \wedge X_i),\end{aligned}$$

using Theorem 5.4. So at the level of homology,

$$\nabla_{GM} = \tilde{\nabla} + \sum_{i=1}^N \chi(X_i \wedge Y_i).$$

□

**Theorem 8.9.** *As connections on  $HP_{\bullet}^{\mathfrak{g}}(A)$ ,  $\nabla_{GM}$  is integrable if and only if  $\tilde{\nabla}$  is integrable.*

*Proof.* By Theorem 8.3,  $\sum_{i=1}^N \chi(X_i \wedge Y_i)$  is a nilpotent operator on  $HP_{\bullet}^{\mathfrak{g}}(A)$ . The previous two results imply that we can apply Proposition 6.8, which gives the conclusion.  $\square$

## 9. INTEGRATING $\nabla_{GM}$ FOR NONCOMMUTATIVE TORI

In this section, we specialize to the noncommutative tori deformation  $\{\mathcal{A}_{t\Theta}\}_{t \in \mathbb{R}}$  for a given  $n \times n$  skew-symmetric real matrix  $\Theta$  and  $J = \mathbb{R}$ . Let  $A$  denote the section algebra of this deformation,  $\nabla = \frac{d}{dt}$  the canonical connection, and  $E = -\delta\nabla$  the 2-cocycle associated to this connection. We view the canonical derivations  $\delta_1, \dots, \delta_n$  as  $C^\infty(J)$ -linear derivations  $A$ .

**Proposition 9.1.** *In the above situation,*

$$E = 2\pi i \sum_{j>k} \theta_{jk} \cdot \delta_j \smile \delta_k.$$

*Proof.* For any  $a_1, a_2 \in A$  with  $\nabla a_1 = \nabla a_2 = 0$ ,

$$E(a_1, a_2) = \nabla(a_1 a_2) = 2\pi i \sum_{j>k} \theta_{jk} \delta_j(a_1) \delta_k(a_2)$$

by Proposition 3.6. Both  $E(a_1, a_2)$  and the right hand side above are continuous and  $C^\infty(J)$ -bilinear. As the  $C^\infty(J)$ -span of  $\ker \nabla$  is dense in  $A$ , the equality

$$E(a_1, a_2) = 2\pi i \sum_{j>k} \theta_{jk} \delta_j(a_1) \delta_k(a_2)$$

holds for all  $a_1, a_2 \in A$ .  $\square$

**Remark 9.2.** One can show that  $[E] \neq 0$  in  $H^2(A, A)$ . Consequently,  $A$  does not possess a connection that is also a derivation. This is consistent with the fact that the isomorphism class of the algebra  $\mathcal{A}_{t\Theta}$  varies as  $t$  varies.

Let  $\mathfrak{g}$  be the Lie algebra generated by  $\delta_1, \dots, \delta_n$ . As  $[\delta_j, \delta_k] = 0$ ,  $\mathfrak{g}$  is an  $n$ -dimensional abelian Lie algebra. Moreover,  $[\nabla, \delta_j] = 0$ , so our deformation is of the type described in the previous section. In particular, we have the connection

$$\tilde{\nabla} = L_{\nabla} + 2\pi i \sum_{j>k} \theta_{jk} \cdot L\{\delta_j, \delta_k\}$$

on the  $\mathfrak{g}$ -invariant complex  $C_{\bullet}^{\mathfrak{g}}(A)$  which descends to a connection on  $HP_{\bullet}^{\mathfrak{g}}(A)$ . By the results of the previous section,  $\tilde{\nabla}$  is a nilpotent perturbation of  $\nabla_{GM}$  on  $HP_{\bullet}^{\mathfrak{g}}(A)$ , and so  $\tilde{\nabla}$  is integrable if and only if  $\nabla_{GM}$  is integrable.

We shall now show that we are not losing anything in passing to  $\mathfrak{g}$ -invariant cyclic homology, in that the canonical map  $HP_{\bullet}(\mathcal{A}_{\Theta}) \rightarrow HP_{\bullet}^{\mathfrak{g}}(\mathcal{A}_{\Theta})$  is a chain homotopy equivalence.

**Theorem 9.3.** *The canonical map  $C_{\text{per}}(\mathcal{A}_{\Theta}) \rightarrow C_{\text{per}}^{\mathfrak{g}}(\mathcal{A}_{\Theta})$  is a chain homotopy equivalence and thus induces an isomorphism  $HP_{\bullet}(\mathcal{A}_{\Theta}) \cong HP_{\bullet}^{\mathfrak{g}}(\mathcal{A}_{\Theta})$*

*Proof.* Recall that for a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we write  $u^\alpha = u_1^{\alpha_1} \dots u_n^{\alpha_n} \in \mathcal{A}_\Theta$ . Thus the span of elements of the form  $(u^{\alpha^1}, \dots, u^{\alpha^m})$  for any  $m$  multi-indices  $\alpha^1, \dots, \alpha^m$  is dense in  $C_m(\mathcal{A}_\Theta)$ . It is easy to see that for  $\omega = (u^{\alpha^1}, \dots, u^{\alpha^m}) \in C_m(\mathcal{A}_\Theta)$ ,

$$L_{\delta_j} \omega = (\deg_j \omega) \omega$$

where  $\deg_j \omega = \sum_{i=1}^m \alpha_j^i$ . Thus,  $\ker L_{\delta_j}$  is just the degree 0 part of  $C_{\text{per}}(\mathcal{A}_\Theta)$  with respect to the grading given by  $\deg_j$ . Define

$$N_j(\omega) = \begin{cases} 0 & \deg_j \omega = 0 \\ \frac{1}{\deg_j \omega} \omega & \deg_j \omega \neq 0 \end{cases}$$

for homogeneous  $\omega \in C_{\text{per}}(\mathcal{A}_\Theta)$ . Using the fact that  $\mathcal{S}(\mathbb{Z}^n)^{\widehat{\otimes} m} \cong \mathcal{S}(\mathbb{Z}^{nm})$  as a topological vector space, it is easy to see that  $N_j$  extends to a continuous operator on  $C_{\text{per}}(\mathcal{A}_\Theta)$ . Moreover,  $p_j := 1 - N_j L_{\delta_j} : C_{\text{per}}(\mathcal{A}_\Theta) \rightarrow \ker L_{\delta_j}$  is the projection.

Let  $M_j = \cap_{k=1}^j \ker L_{\delta_k}$ . Then for  $1 \leq j \leq n$ , consider the inclusion  $i_j : M_j \rightarrow M_{j-1}$  and projection  $p_j : M_{j-1} \rightarrow M_j$ . Both maps are chain maps, and we claim they are inverses up to chain homotopy. It is immediate that  $p_j i_j = 1$ . Let  $h_j = N_j L_{\delta_j}$ . For any  $k$ ,  $L_{\delta_k}$  commutes with  $N_j$  and also with  $L_{\delta_j}$  because

$$[L_{\delta_k}, L_{\delta_j}] = L_{[\delta_k, \delta_j]} = 0.$$

Consequently,  $h_j$  maps  $M_{j-1}$  into  $M_{j-1}$ . Using Theorem 5.3 and the fact that  $[b + B, N_j] = 0$ , we have

$$[b + B, h_j] = N_j [b + B, L_{\delta_j}] = N_j L_{\delta_j} = 1 - i_j p_j.$$

By composing these chain equivalences, we see that the inclusion

$$i : \cap_{k=1}^n \ker L_{\delta_k} = M_n \rightarrow M_0 = C_{\text{per}}(\mathcal{A}_\Theta)$$

is a chain homotopy equivalences with homotopy inverse  $p = p_1 \dots p_n$ .

Now, there is a natural chain map  $\tilde{i} : \cap_{j=1}^n \ker L_{\delta_j} \rightarrow C_{\text{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta)$  which is the composition of the inclusion  $i$  and the natural projection  $\pi : C_{\text{per}}(\mathcal{A}_\Theta) \rightarrow C_{\text{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta)$ . Further, it is not difficult to show that  $p$  factors through  $\pi$  to give a map  $\bar{p} : C_{\text{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta) \rightarrow \cap_{j=1}^n \ker L_{\delta_j}$  which is an inverse chain map to  $\tilde{i}$ . Thus,  $\pi = \bar{p}^{-1} p$  is a chain equivalence as desired because it is the composition of a chain equivalence and an isomorphism of complexes.  $\square$

Note that the same proof shows that for the section algebra  $A$  of the noncommutative tori deformation, the natural map  $C_{\text{per}}(A) \rightarrow C_{\text{per}}^{\mathfrak{g}}(A)$  is a chain equivalence, where the theories are considered over the algebra  $C^\infty(J)$ .

Our goal now is to show that the connection

$$\tilde{\nabla} = L_\nabla + 2\pi i \sum_{j>k} \theta_{jk} \cdot L\{\delta_j, \delta_k\}$$

on the invariant complex  $C_{\text{per}}^{\mathfrak{g}}(A)$  is integrable. We will switch gears and do this on the cohomology side, although we could just as easily do the same for homology.

For an algebra  $A$  over  $R$  with an action of a Lie algebra  $\mathfrak{g}$  by derivations, let  $C_{\mathfrak{g}}^{\text{per}}(A) = \text{Hom}_R(C_{\text{per}}^{\mathfrak{g}}, R)$  be the dual complex. Thus  $C_{\mathfrak{g}}^{\text{per}}(A)$  consists of all linear functionals  $\varphi : C_{\text{per}}(A) \rightarrow R$  such that  $\varphi(L_X \omega) = 0$  for all  $X \in \mathfrak{g}$  and all  $\omega \in C_{\text{per}}(A)$ . The invariant periodic cyclic cohomology groups are denoted  $HP_{\mathfrak{g}}^\bullet(A)$ . In the case of noncommutative tori, by taking the transpose of the chain equivalence

in Theorem 9.3, we see that the natural inclusion  $C_{\mathfrak{g}}^{\text{per}}(\mathcal{A}_{\Theta}) \rightarrow C^{\text{per}}(\mathcal{A}_{\Theta})$  is a chain equivalence. Similarly for the section algebra  $A$ ,  $C_{\mathfrak{g}}^{\text{per}}(A) \rightarrow C^{\text{per}}(A)$  is a chain equivalence over  $C^{\infty}(J)$ .

The dual connection of  $\tilde{\nabla}$ , which we still denote  $\tilde{\nabla}$  is given by

$$(\tilde{\nabla}\varphi)(\omega) = \frac{d}{dt}\varphi(\omega) - \varphi(L_{\nabla}\omega) - 2\pi i \sum_{j>k} \theta_{jk} \varphi(L\{\delta_j, \delta_k\}\omega).$$

**Theorem 9.4.** *For the section algebra  $A$  of the noncommutative tori deformation, the connection  $\tilde{\nabla}$  is integrable on  $C^{\text{per}}(A)$  and consequently on  $C_{\mathfrak{g}}^{\text{per}}(A)$ .*

*Proof.* Since  $\tilde{\nabla}$  restricts to a connection on  $C^m(A)$  for each  $m$ , it suffices to prove  $\tilde{\nabla}$  is integrable on  $C^m(A)$ . Given  $m+1$  multi-indices  $\alpha^0, \dots, \alpha^m$ , each of length  $n$ , we shall use the notation

$$u^{\bar{\alpha}} = (u^{\alpha^0}, u^{\alpha^1}, \dots, u^{\alpha^m}) \in C_m(\mathcal{A}_{\Theta}).$$

By continuity, any  $\psi \in C^m(\mathcal{A}_{\Theta})$  is determined by the numbers  $c_{\bar{\alpha}} := \psi(u^{\bar{\alpha}})$  for  $\bar{\alpha} \in \prod_{i=0}^m \mathbb{Z}^n$ . Such a collection of numbers  $\{c_{\bar{\alpha}}\}$  gives a continuous  $\psi \in C^m(\mathcal{A}_{\Theta})$  if and only if  $|c_{\bar{\alpha}}| \leq P(\bar{\alpha})$  for some polynomial in the indices of  $\bar{\alpha}$ .

Likewise, if  $\varphi \in C_{C^{\infty}(J)}^m(A)$ , then  $\varphi$  is determined by functions  $f_{\bar{\alpha}} := \varphi(u^{\bar{\alpha}}) \in C^{\infty}(J)$ . By Proposition 2.4,  $\varphi$  is given by a smooth family of continuous linear functionals  $\varphi_t \in C^m(\mathcal{A}_{t\Theta})$ . Consequently, for each  $t \in J$ , the numbers  $|f_{\bar{\alpha}}^{(k)}(t)|$  must satisfy the polynomial growth condition above for each integer  $k \geq 0$ .

For a fixed  $t_0 \in J$  and fixed  $\psi \in C^m(\mathcal{A}_{t_0\Theta})$ , we must prove the existence and uniqueness of solutions to the system of differential equations

$$\tilde{\nabla}\varphi = 0, \quad \varphi_{t_0} = \psi.$$

As above, let  $c_{\bar{\alpha}} = \psi(u^{\bar{\alpha}})$  and  $f_{\bar{\alpha}} = \varphi(u^{\bar{\alpha}})$ . The crucial point is that

$$\sum_{j>k} L\{\delta_j, \delta_k\} u^{\bar{\alpha}} = R(\bar{\alpha}) u^{\bar{\alpha}},$$

where  $R(\bar{\alpha})$  is some real-valued polynomial in the multi-indices. Thus,

$$\sum_{j>k} \varphi(L\{\delta_j, \delta_k\} u^{\bar{\alpha}}) = R(\bar{\alpha}) f_{\bar{\alpha}}.$$

Now,  $\tilde{\nabla}\varphi = 0$  if and only if  $(\tilde{\nabla}\varphi)(u^{\bar{\alpha}}) = 0$  for each  $\bar{\alpha}$ . Combined with the initial condition, this says

$$f'_{\bar{\alpha}} - 2\pi i R(\bar{\alpha}) f_{\bar{\alpha}} = 0, \quad f_{\bar{\alpha}}(t_0) = c_{\bar{\alpha}}.$$

This elementary initial value problem has the unique solution

$$f_{\bar{\alpha}}(t) = c_{\bar{\alpha}} \exp(2\pi i R(\bar{\alpha})(t - t_0)).$$

Since  $|\exp(2\pi i R(\bar{\alpha})(t - t_0))| = 1$  and  $R(\bar{\alpha})$  is a polynomial, we see that for each  $t \in J$  and  $k \geq 0$ , the numbers  $|f_{\bar{\alpha}}^{(k)}(t)|$  are of polynomial growth. Thus, the  $f_{\bar{\alpha}}$  do indeed give a unique solution  $\varphi$ , which is an element of  $C^m(A)$ . Using the explicit form of  $\varphi$  given above, one can show that the map  $\psi \mapsto \varphi$  described above is continuous, i.e. the  $\{f_{\bar{\alpha}}\}$  depend continuously on the  $\{c_{\bar{\alpha}}\}$ . So  $\tilde{\nabla}$  is integrable on  $C^{\text{per}}(A)$ .

The fact that  $[L_X, \tilde{\nabla}] = 0$  for all  $X \in \mathfrak{g}$  combined with the above existence and uniqueness result show that if the initial data is in  $C_{\mathfrak{g}}^{\text{per}}(\mathcal{A}_{t\Theta})$ , then the solution is in  $C_{\mathfrak{g}}^{\text{per}}(A)$ . Thus  $\tilde{\nabla}$  is integrable on  $C_{\mathfrak{g}}^{\text{per}}(A)$ .  $\square$

**Corollary 9.5.** *For any  $n \times n$  skew-symmetric matrix  $\Theta$ , there is a continuous chain homotopy equivalence*

$$C^{\text{per}}(\mathcal{A}_{\Theta}) \rightarrow C^{\text{per}}(C^{\infty}(\mathbb{T}^n))$$

and therefore an isomorphism

$$HP^{\bullet}(\mathcal{A}_{\Theta}) \cong HP^{\bullet}(C^{\infty}(\mathbb{T}^n)).$$

Consequently,

$$HP^0(\mathcal{A}_{\Theta}) \cong \mathbb{C}^{2^{n-1}}, \quad HP^1(\mathcal{A}_{\Theta}) \cong \mathbb{C}^{2^{n-1}}.$$

*Proof.* The chain homotopy equivalence is the composition

$$C^{\text{per}}(\mathcal{A}_{\Theta}) \rightarrow C_{\mathfrak{g}}^{\text{per}}(\mathcal{A}_{\Theta}) \xrightarrow{P^{\tilde{\nabla}}} C_{\mathfrak{g}}^{\text{per}}(\mathcal{A}_0) \rightarrow C^{\text{per}}(\mathcal{A}_0),$$

where the first and last arrows are the chain equivalences from Theorem 9.3. Since  $\tilde{\nabla}$  is a chain map on  $C_{\mathfrak{g}}^{\text{per}}(A)$ , the middle arrow is an isomorphism of complexes by Proposition 6.12.

As shown in [3], if  $M$  is a compact smooth manifold, then

$$HP^{\bullet}(C^{\infty}(M)) \cong \bigoplus_k H_{\bullet+2k}^{dR}(M, \mathbb{C}),$$

where  $H_{\bullet}^{dR}(M, \mathbb{C})$  is the complex-valued de Rham homology of  $M$ . Now,  $H_m^{dR}(\mathbb{T}^n, \mathbb{C})$  is a vector space of dimension  $\binom{n}{m}$ , and this gives the result.  $\square$

We have proved the rigidity of periodic cyclic cohomology for the deformation of noncommutative tori. It is interesting to note that the Hochschild cohomology  $H^{\bullet}(\mathcal{A}_{\Theta})$  and (non periodic) cyclic cohomology  $HC^{\bullet}(\mathcal{A}_{\Theta})$  are very far from rigid in this deformation. As an example,  $H^0(A) = HC^0(A)$  is the space of all traces on the algebra  $A$ . Now in the simplest case where  $n = 2$  and  $\Theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ , it is well-known that there is a unique (normalized) trace on  $\mathcal{A}_{\Theta}$  when  $\theta \notin \mathbb{Q}$  and an infinite dimensional space of traces when  $\theta \in \mathbb{Q}$ . For example, every linear functional on the commutative algebra  $\mathcal{A}_0 \cong C^{\infty}(\mathbb{T}^n)$  is a trace, and thus  $H^0(C^{\infty}(\mathbb{T}^n)) = C^{\infty}(\mathbb{T}^n)^*$  is the space of distributions on  $\mathbb{T}^n$ . Moreover, Connes showed in [3] that in the case  $\theta \notin \mathbb{Q}$ ,  $H^1(\mathcal{A}_{\Theta})$  and  $H^2(\mathcal{A}_{\Theta})$  are either finite dimensional or infinite dimensional and non-Hausdorff depending on the diophantine properties of  $\theta$ . Looking back, we conclude that there are no integrable connections on  $C^{\bullet}(A)$  that commute with  $b$ , as such a connection would imply rigidity of Hochschild cohomology.

However, our connection  $\tilde{\nabla}$  does commute with  $b$  on the invariant complex  $C_{\mathfrak{g}}^{\bullet}(A)$ . This shows that the invariant Hochschild cohomology  $H_{\mathfrak{g}}^{\bullet}(\mathcal{A}_{\Theta})$  is independent of  $\Theta$ . For example, there is exactly one (normalized)  $\mathfrak{g}$ -invariant trace on  $C^{\infty}(\mathbb{T}^n)$ , and that corresponds to integration with respect to the only (normalized) translation invariant measure. Thus  $H_{\mathfrak{g}}^0(C^{\infty}(\mathbb{T}^n)) = H_{\mathfrak{g}}^0(\mathcal{A}_{\Theta}) = \mathbb{C}$ . Consequently, the canonical map  $H_{\mathfrak{g}}^{\bullet}(\mathcal{A}_{\Theta}) \rightarrow H^{\bullet}(\mathcal{A}_{\Theta})$  is not, in general, an isomorphism.

### 9.1. Variations.

9.1.1. *Algebraic noncommutative tori.* One can consider the subalgebra  $\mathcal{A}_\Theta^{\text{alg}} \subset \mathcal{A}_\Theta$  generated by  $u_1, \dots, u_n, u_1^{-1}, \dots, u_n^{-1}$  as an algebra without topology. With regards to the (algebraic) periodic cyclic cohomology, we still have a chain equivalence  $HP_\bullet^\bullet(\mathcal{A}_\Theta^{\text{alg}}) \rightarrow HP^\bullet(\mathcal{A}_\Theta^{\text{alg}})$ , and the parallel transport isomorphism  $P^{\tilde{\nabla}}$  extends in the obvious way to an isomorphism

$$P^{\tilde{\nabla}} : HP_\bullet^\bullet(\mathcal{A}_\Theta^{\text{alg}}) \rightarrow HP_\bullet^\bullet(\mathcal{A}_0^{\text{alg}}),$$

where  $\mathcal{A}_0^{\text{alg}} \cong \mathbb{C}[u_1, \dots, u_n, u_1^{-1}, \dots, u_n^{-1}]$ .

9.1.2. *Entire cyclic cohomology.* Entire cyclic cohomology is a variation of periodic cyclic cohomology introduced by Connes in [1] which allows for cochains of infinite support that decay sufficiently fast. Recall that a subset  $\Sigma$  of a locally convex vector space  $X$  is *bounded* if for every seminorm  $p$  defining the topology,

$$\sup_{x \in \Sigma} p(x) < \infty.$$

An *even entire cochain* is an element  $\varphi = (\varphi_{2n}) \in \prod_{n=0}^\infty C^{2n}(A)$  such that for every bounded subset  $\Sigma \subset A$  and every real  $\lambda > 0$ , there exists a constant  $C = C_{\Sigma, \lambda}$  such that for all  $n \geq 0$ ,

$$|\varphi_{2n}(a_0, \dots, a_{2n})| \leq C \frac{\lambda^n}{n!}$$

for all  $a_i \in \Sigma$ . Likewise, an *odd entire cochain*  $\varphi = (\varphi_{2n+1}) \in \prod_{n=0}^\infty C^{2n+1}(A)$  is such that there exists  $C = C_{\Sigma, \lambda}$  so that

$$|\varphi_{2n+1}(a_0, \dots, a_{2n+1})| \leq C \frac{\lambda^n}{n!}$$

for all  $a_i \in \Sigma$ . Let  $C_\epsilon^{\text{even}}(A)$  and  $C_\epsilon^{\text{odd}}(A)$  denote the spaces of even and odd entire cochains and let  $C_\epsilon^{\text{per}}(A) = C_\epsilon^{\text{even}}(A) \oplus C_\epsilon^{\text{odd}}(A)$ . One can show that the operators  $b$  and  $B$  map entire cochains to entire cochains, and so  $C_\epsilon^{\text{per}}(A)$  is a  $\mathbb{Z}/2$ -graded complex with differential  $b + B$ . Its cohomology is the *entire cyclic cohomology* of  $A$ , which we shall denote  $HE^\bullet(A)$ . As  $C^{\text{per}}(A)$  is a subcomplex of  $C_\epsilon^{\text{per}}(A)$ , there is a natural map

$$HP^\bullet(A) \rightarrow HE^\bullet(A)$$

which is, in general, neither injective nor surjective. The following result is due to Michael Puschnigg [12].

**Theorem 9.6.** *If  $M$  is a compact smooth manifold, then the natural map*

$$HP^\bullet(C^\infty(M)) \rightarrow HE^\bullet(C^\infty(M))$$

*is an isomorphism.*

Combining our work with this result, we obtain

**Theorem 9.7.** *The canonical map*

$$HP^\bullet(\mathcal{A}_\Theta) \rightarrow HE^\bullet(\mathcal{A}_\Theta)$$

*is an isomorphism.*



*Proof.* Using the fact that continuous linear maps send bounded sets to bounded sets, it is not difficult to show that the operators  $L_D, I_D, L\{X, Y\}, I\{X, Y\}$  map entire cochains to entire cochains. Thus, all homotopy formulas involving these operators extend to the entire cyclic complex. For the noncommutative torus  $\mathcal{A}_\Theta$ , we can introduce the  $\mathfrak{g}$ -invariant entire cochain complex and  $\mathfrak{g}$ -invariant entire cyclic cohomology  $HE_\mathfrak{g}^\bullet(\mathcal{A}_\Theta)$  as before. The results of this section carry over in the same way to the entire case. In particular, the chain equivalence

$$P : C^{\text{per}}(\mathcal{A}_\Theta) \rightarrow C^{\text{per}}(C^\infty(\mathbb{T}^n))$$

extends to a chain equivalence

$$P_\epsilon : C_\epsilon^{\text{per}}(\mathcal{A}_\Theta) \rightarrow C_\epsilon^{\text{per}}(C^\infty(\mathbb{T}^n))$$

such that the diagram

$$C^{\text{per}}(\mathcal{A}_\Theta) \xrightarrow{P} C^{\text{per}}(C^\infty(\mathbb{T}^n))$$

$$C_\epsilon^{\text{per}}(\mathcal{A}_\Theta) \xrightarrow{P_\epsilon} C_\epsilon^{\text{per}}(C^\infty(\mathbb{T}^n))$$

commutes. Consequently, the induced commutative diagram

$$\begin{array}{ccc} HP^\bullet(\mathcal{A}_\Theta) & \xrightarrow{\cong} & HP^\bullet(C^\infty(\mathbb{T}^n)) \\ & & \cong \\ HE^\bullet(\mathcal{A}_\Theta) & \xrightarrow{\cong} & HE^\bullet(C^\infty(\mathbb{T}^n)) \end{array}$$

gives the result.  $\square$

## 10. DIFFERENTIATION FORMULAS FOR CYCLIC COCYCLES IN NONCOMMUTATIVE TORI

Recall that a (simplicially normalized) *cyclic cocycle*  $\varphi \in C^k(A)$  is a Hochschild cocycle such that

$$\varphi(a_k, a_0, \dots, a_{k-1}) = (-1)^k \varphi(a_0, a_1, \dots, a_k).$$

A cyclic cocycle  $\varphi$  automatically satisfies  $B\varphi = 0$ , because

$$\varphi(1, a_0, \dots, a_{k-1}) = (-1)^k \varphi(a_{k-1}, 1, a_0, \dots, a_{k-2}) = 0$$

by definition of  $C^k(A)$ . Thus,  $(b + B)\varphi = 0$ , and so  $\varphi$  gives a cohomology class in  $HP^\bullet(A)$

Suppose that  $\mathfrak{g}$  is an abelian Lie algebra of derivations on an algebra  $A$ , and suppose  $\tau$  is a trace which is  $\mathfrak{g}$ -invariant in the sense that 1

$$\tau \circ X = 0, \quad \forall X \in \mathfrak{g}.$$

Define the *characteristic map*  $\gamma : \Lambda^\bullet \mathfrak{g} \rightarrow C^\bullet(A)$  by

$$\gamma(X_1 \wedge \dots \wedge X_k)(a_0, \dots, a_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} (-1)^\sigma \tau(a_0 X_{\sigma(1)}(a_1) X_{\sigma(2)}(a_2) \dots X_{\sigma(k)}(a_k)).$$

**Proposition 10.1.** *The functional  $\gamma(X_1 \wedge \dots \wedge X_k)$  is a cyclic  $k$ -cocycle. Moreover, it is invariant with respect to the action of  $\mathfrak{g}$  by Lie derivatives.*

**Remark 10.2.** The map  $\gamma$  is a simple case of the Connes-Moscovici characteristic map in Hopf cyclic cohomology [2]. In their work,  $\mathcal{H}$  is a Hopf algebra equipped with some extra structure called a modular pair, and  $A$  is an algebra equipped with a Hopf action of  $\mathcal{H}$ . Assuming  $A$  possesses a trace that is compatible with the modular pair, they construct a map

$$\gamma : HP_{Hopf}^\bullet(\mathcal{H}) \rightarrow HP^\bullet(A)$$

from the Hopf periodic cyclic cohomology of  $\mathcal{H}$  to the ordinary periodic cyclic cohomology of  $A$ . In our situation,  $\mathcal{H} = \mathcal{U}(\mathfrak{g})$ , the universal enveloping algebra of  $\mathfrak{g}$ . The fact that  $\mathfrak{g}$  acts on  $A$  by derivations implies that the action of  $\mathcal{U}(\mathfrak{g})$  on  $A$  is a Hopf action. The compatibility condition for the trace follows from the fact that our trace is  $\mathfrak{g}$ -invariant. As was shown in [2],

$$HP_{Hopf}^\bullet(\mathcal{U}(\mathfrak{g})) \cong \bigoplus_{k=\bullet \bmod 2} H_k^{Lie}(\mathfrak{g}, \mathbb{C}),$$

where  $H_k^{Lie}(\mathfrak{g}, \mathbb{C})$  is the Lie algebra homology of  $\mathfrak{g}$  with coefficients in the trivial  $\mathfrak{g}$ -module  $\mathbb{C}$ . As  $\mathfrak{g}$  is abelian, there is an isomorphism

$$H_k^{Lie}(\mathfrak{g}, \mathbb{C}) \cong \Lambda^k(\mathfrak{g}).$$

The obtained characteristic map

$$\gamma : \Lambda^\bullet(\mathfrak{g}) \rightarrow HP^\bullet(A)$$

is the map defined above.

The fact that  $\gamma(X_1 \wedge \dots \wedge X_k)$  is invariant relies on the fact that  $\mathfrak{g}$  is abelian. In this case, the characteristic map factors through the inclusion  $HP_{\mathfrak{g}}^\bullet(A) \rightarrow HP^\bullet(A)$ , and we obtain a characteristic map

$$\gamma : \Lambda^\bullet(\mathfrak{g}) \rightarrow HP_{\mathfrak{g}}^\bullet(A).$$

**Lemma 10.3.** *Let  $X_1, \dots, X_n$  be derivations on an algebra  $A$ , and let  $\tau$  be a trace on  $A$ . There exists  $\psi \in C^{n-1}(A)$  such that  $B\psi = 0$  and*

$$\begin{aligned} & \tau(a_0 X_1(a_1) \dots X_n(a_n)) \\ &= \frac{1}{n} \sum_{j=1}^n (-1)^{(j-1)(n+1)} \tau(a_0 X_j(a_1) \dots X_n(a_{n-j+1}) X_1(a_{n-j+2}) \dots X_{j-1}(a_n)) \\ & \quad + (b\psi)(a_0, \dots, a_n). \end{aligned}$$

*Proof.* Given any  $n$  derivations  $Y_1, \dots, Y_n$ , the  $(n-1)$ -cochain

$$\varphi(a_0, \dots, a_{n-1}) = \tau(Y_1(a_0) Y_2(a_1) \dots Y_n(a_{n-1}))$$

satisfies

$$(b\varphi)(a_0, \dots, a_n) = \tau(a_0 Y_1(a_1) \dots Y_n(a_n)) + (-1)^n a_0 Y_2(a_1) \dots Y_n(a_{n-1}) Y_1(a_n)$$

and  $B\varphi = 0$ . It follows that

$$\begin{aligned} & \psi(a_0, \dots, a_{n-1}) \\ &= \frac{1}{n} \sum_{j=1}^{n-1} (-1)^{(j-1)(n+1)} (n-j) \tau(X_j(a_0) X_{j+1}(a_1) \dots X_{j-1}(a_n)) \end{aligned}$$

satisfies the conclusions of the lemma.  $\square$

Recall that for any  $Z \in \mathfrak{g}$ , the cyclic contraction  $I_Z$  is a chain map on the invariant complex  $C_{\mathfrak{g}}^{\text{per}}(A)$ .

**Proposition 10.4.** *For any  $Z, X_1, \dots, X_k \in \mathfrak{g}$ ,*

$$I_Z[\gamma(X_1 \wedge \dots \wedge X_k)] = [\gamma(Z \wedge X_1 \wedge \dots \wedge X_k)]$$

*in  $HP_{\mathfrak{g}}^{\bullet}(A)$ .*

*Proof.* Let  $\varphi = \gamma(X_1 \wedge \dots \wedge X_k)$ . Since  $\varphi$  is cyclic and normalized,

$$\varphi(1, a_1, \dots, a_k) = 0, \quad a_1, \dots, a_k \in A,$$

and consequently  $S_Z \varphi = 0$ . Thus,  $I_Z \varphi = \iota_Z \varphi$ , and

$$\begin{aligned} (\iota_Z \varphi)(a_0, \dots, a_{k+1}) &= \varphi(a_0 Z(a_1), a_2, \dots, a_{k+1}) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} (-1)^{\sigma} \tau(a_0 Z(a_1) X_{\sigma(1)}(a_2) X_{\sigma(2)}(a_3) \dots X_{\sigma(k)}(a_{k+1})) \\ &= \gamma(Z \wedge X_1 \wedge \dots \wedge X_k)(a_0, \dots, a_{k+1}) + (b\psi)(a_0, \dots, a_{k+1}) \end{aligned}$$

for some  $\psi$  with  $B\psi = 0$  by applying the previous lemma to each term in the sum. Hence,  $I_Z \gamma(X_1 \wedge \dots \wedge X_k) = \gamma(Z \wedge X_1 \wedge \dots \wedge X_k) + (b + B)\psi$ .  $\square$

As in the homology case (Theorem 8.3,) there is an algebra map

$$\chi : \Lambda^{\bullet}(\mathfrak{g}) \rightarrow \text{End}(HP_{\mathfrak{g}}^{\bullet}(A))$$

given by

$$\chi(X_1 \wedge \dots \wedge X_k) = I_{X_1} I_{X_2} \dots I_{X_k}.$$

**Corollary 10.5.** *For any  $X_1, \dots, X_k \in \mathfrak{g}$ ,*

$$[\gamma(X_1 \wedge \dots \wedge X_k)] = \chi(X_1 \wedge \dots \wedge X_k)[\tau]$$

*in  $HP_{\mathfrak{g}}^{\bullet}(A)$ .*

**Remark 10.6.** A generalization of the Connes-Moscovici characteristic map was constructed in [9]. A special case of this construction is a cup product

$$\smile : HP_{\text{Hof}}^p(\mathcal{H}) \otimes HP_{\mathcal{H}}^q(A) \rightarrow HP^{p+q}(A),$$

where  $HP_{\mathcal{H}}^{\bullet}(A)$  is the periodic cyclic cohomology of  $A$  built out of cochains which are invariant in some sense with respect to the action of  $\mathcal{H}$ . In the Connes-Moscovici picture, the properties of the trace  $\tau$  ensures that it gives a cohomology class in  $HP_{\mathcal{H}}^{\bullet}(A)$ , and

$$[\omega] \smile [\tau] = \gamma[\omega]$$

for all  $[\omega] \in HP_{\text{Hof}}^{\bullet}(\mathcal{H})$ . In our situation where  $\mathcal{H} = \mathcal{U}(\mathfrak{g})$ , we have that  $HP_{\mathcal{H}}^{\bullet}(A) = HP_{\mathfrak{g}}^{\bullet}(A)$  and the cup product will give a map

$$\smile : \Lambda^p \mathfrak{g} \otimes HP_{\mathfrak{g}}^q(A) \rightarrow HP^{p+q}(A).$$

Our map  $\chi : \Lambda^{\bullet}(\mathfrak{g}) \rightarrow \text{End}(HP_{\mathfrak{g}}^{\bullet}(A))$  followed by the canonical inclusion  $HP_{\mathfrak{g}}^{\bullet}(A) \rightarrow HP^{\bullet}(A)$  coincides with this cup product. The fact that the cup product sends  $\mathfrak{g}$ -invariant cocycles to  $\mathfrak{g}$ -invariant cocycles is a consequence of the fact that  $\mathfrak{g}$  is abelian.

**10.1. Noncommutative tori.** Now, let  $A$  be the section algebra of the noncommutative  $n$ -tor deformation  $\{\mathcal{A}_{t\Theta}\}_{t \in J}$ . More generally, we shall consider the matrix algebra  $M_N(A)$ , which is the section algebra of the deformation  $\{M_N(\mathcal{A}_{t\Theta})\}_{t \in J}$ . Using the isomorphism  $M_N(A) \cong M_N(\mathbb{C}) \otimes A$ , the operators  $\text{tr} \otimes \tau$ ,  $1 \otimes \delta_j$ , and  $1 \otimes \nabla$  shall be denoted by  $\tau$ ,  $\delta_j$ , and  $\nabla$ . Then just as in the  $N = 1$  case, one has

$$E := -\delta \nabla = 2\pi i \sum_{j>k} \theta_{jk} \cdot \delta_j \smile \delta_k.$$

The map  $T : C^\bullet(A) \rightarrow C^\bullet(M_N(A))$ , which is the transpose of the generalized trace, induces an isomorphism  $HP^\bullet(A) \rightarrow HP^\bullet(M_N(A))$  which is parallel with respect to the Gauss-Manin connections. Thus,  $\nabla^{GM}$  is integrable on  $HP^\bullet(M_N(A))$ .

**Proposition 10.7.** *For any  $X_1, \dots, X_m \in \mathfrak{g}$ ,*

$$\widetilde{\nabla}(\gamma(X_1 \wedge \dots \wedge X_m)) = 0.$$

*Proof.* This is proved by noticing  $\frac{d}{dt} \circ \tau = \tau \circ \nabla$  and

$$\begin{aligned} \nabla(a_0 \dots a_m) &= \sum_{j=0}^k a_0 \dots \nabla(a_j) \dots a_m \\ &\quad + 2\pi i \sum_{j' < k'} \sum_{j > k} \theta_{jk} \cdot a_0 \dots \delta_j(a_{j'}) \dots \delta_k(a_{k'}) \dots a_m \end{aligned}$$

and using the fact that  $\mathfrak{g}$  is abelian.  $\square$

**Theorem 10.8.** *For every  $\Theta$ , the map  $\gamma : \Lambda^\bullet(\mathfrak{g}) \rightarrow HP^\bullet(M_N(\mathcal{A}_\Theta))$  is an isomorphism of  $\mathbb{Z}/2$ -graded spaces.*

*Proof.* It suffices to prove the  $N = 1$  case as the general case follows by the fact that the diagram

$$\begin{array}{ccc} \Lambda^\bullet \mathfrak{g} & \xrightarrow{\gamma} & HP^\bullet(\mathcal{A}_\Theta) \\ & \searrow \gamma & \uparrow T \\ & & HP^\bullet(M_N(\mathcal{A}_\Theta)) \end{array}$$

commutes. By Theorem 9.4 and Proposition 10.7, it suffices to prove this for  $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$ . Let  $s_1, \dots, s_n$  be the coordinates in  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ . Choosing a subset of coordinates  $s_{i_1}, \dots, s_{i_m}$  determines a subtorus  $T$  of dimension  $m$ . All such subtori are in bijection with the homology classes of  $\mathbb{T}^n$ . The de Rham cycle corresponding to  $T$  is given by integration of a differential form over  $T$ . The cochain in  $C^m(C^\infty(\mathbb{T}^n))$  corresponding to this cycle is

$$\varphi_T(f_0, \dots, f_m) = \int_T f_0 df_1 \wedge \dots \wedge df_m,$$

and one can show  $\varphi_T = \gamma(\delta_{i_1} \wedge \dots \wedge \delta_{i_m})$  up to a scalar multiple.  $\square$

**Theorem 10.9.** *For any  $X_1, \dots, X_m \in \mathfrak{g}$ ,*

$$\nabla^{GM}[\gamma(X_1 \wedge \dots \wedge X_m)] = 2\pi i \sum_{j>k} \theta_{jk} \cdot [\gamma(\delta_j \wedge \delta_k \wedge X_1 \wedge \dots \wedge X_m)].$$

*Proof.* The analogue of Proposition 8.8 in this setting is that

$$\nabla^{GM} = \tilde{\nabla} + 2\pi i \sum_{j>k} \theta_{jk} \cdot \chi(\delta_j \wedge \delta_k)$$

as operators on  $HP_{\mathfrak{g}}^{\bullet}(A)$ . The result is immediate from Proposition 10.7 and Corollary 10.5.  $\square$

**Corollary 10.10.** *For any idempotent  $P \in M_N(A)$  and  $X_1, \dots, X_{2m} \in \mathfrak{g}$ ,*

$$\begin{aligned} \frac{d}{dt} \gamma(X_1 \wedge \dots \wedge X_{2m})(P, \dots, P) \\ = -2\pi i (4m+2) \sum_{j>k} \theta_{jk} \cdot \gamma(\delta_j \wedge \delta_k \wedge X_1 \wedge \dots \wedge X_{2m})(P, \dots, P). \end{aligned}$$

**Corollary 10.11.** *For any invertible  $U \in M_N(A)$  and  $X_1, \dots, X_{2m+1} \in \mathfrak{g}$ ,*

$$\begin{aligned} \frac{d}{dt} \gamma(X_1 \wedge \dots \wedge X_{2m+1})(U^{-1}, U, \dots, U^{-1}, U) \\ = -2\pi i m \sum_{j>k} \theta_{jk} \cdot \gamma(\delta_j \wedge \delta_k \wedge X_1 \wedge \dots \wedge X_{2m+1})(U^{-1}, U, \dots, U^{-1}, U). \end{aligned}$$

To prove the corollaries, we use the formula

$$\frac{d}{dt} \langle [\varphi], [\omega] \rangle = \langle \nabla^{GM}[\varphi], [\omega] \rangle + \langle [\varphi], \nabla_{GM}[\omega] \rangle$$

for all  $[\varphi] \in HP^{\bullet}(M_N(A))$ ,  $[\omega] \in HP_{\bullet}(M_N(A))$  combined with Corollaries 7.4 and 7.5 and the explicit form of Chern characters.

Let us specialize to the case  $n = 2$ . Here, the noncommutative torus is determined by a single real parameter  $\theta := \theta_{21}$ , and we shall denote the algebra by  $\mathcal{A}_{\theta}$ . In the case  $\theta \notin \mathbb{Q}$ ,  $\mathcal{A}_{\theta}$  is also known as (the smooth version of) the irrational rotation algebra. We shall consider  $\{\mathcal{A}_{\theta}\}_{\theta \in J}$  as a smooth one-parameter deformation, where  $J \subset \mathbb{R}$  is an open interval containing 0. Let  $A$  be the section algebra and let  $\tau_2$  be the cyclic 2-cocycle  $\tau_2 = 4\pi i \cdot \gamma(\delta_1 \wedge \delta_2)$ , which is given explicitly<sup>4</sup> by

$$\tau_2(a_0, a_1, a_2) = 2\pi i \cdot \tau(a_0 \delta_1(a_1) \delta_2(a_2) - a_0 \delta_2(a_1) \delta_1(a_2)).$$

Corollary 10.10 says that for any idempotent  $P \in M_N(A)$ ,

$$\frac{d}{d\theta} \tau(P) = \tau_2(P, P, P),$$

and

$$\frac{d^2}{d\theta^2} \tau(P) = \frac{d}{d\theta} \tau_2(P, P, P) = 0$$

because  $\gamma(\delta_2 \wedge \delta_1 \wedge \delta_2 \wedge \delta_1) = 0$ . Thus we have shown that for any idempotent  $P \in M_N(A)$ ,

$$\tau(P) = \tau(P)(0) + \tau_2(P, P, P)(0) \cdot \theta.$$

Now the idempotent  $P(0) \in M_N(\mathcal{A}_0) \cong M_N(C^{\infty}(\mathbb{T}^2))$  corresponds to a smooth vector bundle over  $\mathbb{T}^2$  and the value  $\tau(P)(0)$  is the dimension of this bundle. The

<sup>4</sup>Warning: Some authors use the derivations  $\partial_j = 2\pi i \cdot \delta_j$ , in which case

$$\tau_2(a_0, a_1, a_2) = \frac{1}{2\pi i} \tau(a_0 \partial_1(a_1) \partial_2(a_2) - a_0 \partial_2(a_1) \partial_1(a_2)).$$

number  $\tau_2(P, P, P)(0)$  is the first Chern number of the bundle, which is an integer. So  $P$  satisfies

$$\tau(P) = C + D\theta$$

for integers  $C$  and  $D$ .

This result suggests that  $\mathcal{A}_\theta$  may contain an idempotent of trace  $\theta$ , a fact which is now well-known and was first shown in [13]. Let  $P_\theta \in \mathcal{A}_\theta$  be such an idempotent. One could ask about the possibility of extending  $P_\theta$  to an idempotent  $P \in A$ . This is not possible, because such an idempotent would necessarily satisfy  $\tau_2(P, P, P) \neq 0$ , and the only idempotents in  $\mathcal{A}_0$ , namely 0 and 1, do not. However, this can be done in the situation where the parameter space  $J$  doesn't contain any integers.

One can show that there exists an idempotent  $P \in M_2(A)$  of trace  $1 + \theta$  in the case where  $J$  is a small enough interval containing 0. However, this type of phenomenon cannot happen in the following two situations.

- (1) If the parameter space  $J = \mathbb{R}$ , then for any idempotent  $P \in M_N(A)$ , we necessarily have  $\tau(P)$  is a constant integer-valued function. If not,

$$\tau(P) = C + D\theta$$

for some nonzero  $D$ . This contradicts the fact that  $\tau(P)(k) \geq 0$  for all integers  $k$  because  $\mathcal{A}_k \cong C^\infty(\mathbb{T}^2)$ .

- (2) Since the deformation is periodic with period 1, we can consider the parameter space to be  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . In this case, the section algebra has underlying space  $C^\infty(\mathbb{T}, \mathcal{S}(\mathbb{Z}^2))$ , in which multiplication defined in the usual way. The trace is now a map  $\tau : A \rightarrow C^\infty(\mathbb{T})$ . In this case, every idempotent  $P \in M_N(A)$  has constant integer-valued trace. Indeed, by the above results,  $\tau(P)$  must be smooth and locally a linear polynomial in  $\theta$ . The only such functions in  $C^\infty(\mathbb{T})$  are constant.

In a similar fashion, one can consider the cyclic 1-cocycles

$$\tau_1^1 = \gamma(\delta_1), \quad \tau_1^2 = \gamma(\delta_2).$$

For any invertible  $U \in M_N(A)$ ,

$$\frac{d}{d\theta} \tau_1^j(U^{-1}, U) = 0, \quad j = 1, 2$$

by Corollary 10.11, and one can show that  $\tau_1^j(U^{-1}, U)(0)$  is integer-valued. Thus,  $\tau_1^j(U^{-1}, U)$  is a constant integer-valued function.

## APPENDIX A. OMITTED FUNCTIONAL ANALYTIC PROOFS

This appendix contains proofs that were omitted from §§2-3.

We shall use the following lemma which is well-known in the case  $X = \mathbb{R}$  or  $\mathbb{C}$ . It can be proved in the same way.

**Lemma A.1.** *Let  $X$  be a locally convex Hausdorff topological vector space and let  $J \subset \mathbb{R}$  be an open interval. If  $f_n : J \rightarrow X$  are a sequence of continuously differentiable functions that converge pointwise to  $f$ , and if  $f'_n$  converge uniformly on compact sets to a function  $g$ , then  $f' = g$ . In particular, if the series  $h = \sum h_n$  is absolutely convergent and  $\sum h'_n$  converges absolutely and uniformly on compact sets, then  $h' = \sum h'_n$ .*

**Proposition 2.5.** *If  $X$  is barreled, then  $\{F_t : X \rightarrow Y\}_{t \in J}$  is a smooth family of continuous linear maps if and only if the map*

$$t \mapsto F_t(x)$$

*is smooth for every  $x \in X$ .*

*Proof.* Given a set of maps  $\{F_t : X \rightarrow Y\}_{t \in J}$  such that  $t \mapsto F_t(x)$  is smooth for all  $x \in X$ , we must show that the map  $F : X \rightarrow C^\infty(J, Y)$  defined by

$$F(x)(t) = F_t(x)$$

is continuous. For any compact  $K \subset J$ , any natural number  $n$ , and any  $x \in X$ , the set

$$\{F_t^{(n)}(x) | t \in K\}$$

is compact in  $Y$ , hence bounded. By the Banach-Steinhaus theorem, the set  $\{F_t^{(n)}\}_{t \in K}$  is equicontinuous. Thus for any continuous seminorm  $q$  on  $Y$ , there exists a continuous seminorm  $p$  on  $X$  such that

$$q(F_t^{(n)}(x)) \leq p(x), \quad \forall t \in K.$$

Consequently,

$$\sup_{t \in K} q(F_t^{(n)}(x)) \leq p(x),$$

which shows  $F$  is continuous.  $\square$

**Proposition 3.3.** *If  $X$  is a Fréchet space, then a set of continuous associative multiplications  $\{m_t : X \hat{\otimes} X \rightarrow X\}_{t \in J}$  is a smooth one-parameter deformation if and only if the map*

$$t \mapsto m_t(x_1, x_2)$$

*is smooth for each fixed  $x_1, x_2 \in X$ .*

*Proof.* If  $\{m_t\}_{t \in J}$  is a smooth one-parameter deformation, then it is immediate that  $t \mapsto m_t(x_1, x_2)$  is smooth for all  $x_1, x_2 \in X$ .

Conversely, if  $t \mapsto m_t(x_1, x_2)$  is smooth for each fixed  $x_1, x_2 \in X$ , then the map

$$m : X \times X \rightarrow C^\infty(J, X)$$

given by

$$m(x_1, x_2)(t) = m_t(x_1, x_2)$$

is separately continuous by Proposition 2.5. Since  $X$  is Fréchet,  $m$  is jointly continuous, and induces a continuous linear map

$$m : X \hat{\otimes} X \rightarrow C^\infty(J, X).$$

This shows that  $\{m_t\}_{t \in J}$  is a smooth family of continuous linear maps.  $\square$

**Proposition 3.6.** *Given an  $n \times n$  skew-symmetric real matrix  $\Theta$ , the deformation  $\{\mathcal{A}_{t\Theta}\}_{t \in \mathbb{R}}$  is smooth, and for  $x, y$  in the underlying space  $\mathcal{S}(\mathbb{Z}^n)$ ,*

$$\frac{d}{dt} m_t(x, y) = 2\pi i \sum_{j>k} \theta_{jk} m_t(\delta_j(x), \delta_k(y)).$$

*Proof.* By Proposition 3.3, it suffices to show that  $t \mapsto m_t(x, y)$  is smooth for each fixed  $x, y \in \mathcal{S}(\mathbb{Z}^n)$ . First, we will show that it is continuous. Every element  $x \in \mathcal{S}(\mathbb{Z}^n)$  can be expressed as an absolutely convergent series in the natural basis  $\{u_\alpha | \alpha \in \mathbb{Z}^n\}$ . That is, given  $x = \sum_{\alpha \in \mathbb{Z}^n} x_\alpha u^\alpha$ , then

$$\sum_{\alpha \in \mathbb{Z}^n} p_k(x_\alpha u^\alpha) = p_k(x) < \infty$$

for all  $k$ , which says that the series is absolutely convergent in  $\mathcal{S}(\mathbb{Z}^n)$ . So, for fixed  $x, y \in \mathcal{S}(\mathbb{Z}^n)$ , the inequality

$$p_k(m_t(x, y)) \leq p_k(x)p_k(y) < \infty$$

says that the absolute convergence of the series defining  $m_t(x, y)$  is uniform in  $t$ . Since each partial sum in  $m_t(x, y)$  is clearly continuous in  $t$ , the function  $t \mapsto m_t(x, y)$  is continuous.

Next, note that for a fixed  $\alpha$ ,

$$\begin{aligned} \sum_{\beta \in \mathbb{Z}^n} \left| \frac{d}{dt} e^{2\pi i B_\Theta(\alpha - \beta, \beta)t} x_{\alpha - \beta} y_\beta \right| &= \sum_{\beta \in \mathbb{Z}^n} |2\pi i B_\Theta(\alpha - \beta, \beta)| |x_{\alpha - \beta}| |y_\beta| \\ &\leq \sum_{\beta \in \mathbb{Z}^n} q(\beta) |y_\beta| < \infty \end{aligned}$$

for some polynomial  $q(\beta)$ . Here we have used the fact that  $|x_{\alpha - \beta}|$  is bounded and  $B_\Theta(\alpha - \beta, \beta)$  is a polynomial function of  $\beta$ . By Lemma A.1, we have that

$$\frac{d}{dt} [m_t(x, y)_\alpha] = \sum_{\beta \in \mathbb{Z}^n} 2\pi i B_\Theta(\alpha - \beta, \beta) e^{2\pi i B_\Theta(\alpha - \beta, \beta)t} x_{\alpha - \beta} y_\beta.$$

By iterating and using the same argument,

$$\frac{d^r}{dt^r} [m_t(x, y)_\alpha] = \sum_{\beta \in \mathbb{Z}^n} (2\pi i B_\Theta(\alpha - \beta, \beta))^r e^{2\pi i B_\Theta(\alpha - \beta, \beta)t} x_{\alpha - \beta} y_\beta$$

for any positive integer  $r$ . Define the continuous linear map

$$\tilde{E} : \mathcal{S}(\mathbb{Z}^n) \hat{\otimes} \mathcal{S}(\mathbb{Z}^n) \rightarrow \mathcal{S}(\mathbb{Z}^n) \hat{\otimes} \mathcal{S}(\mathbb{Z}^n)$$

by

$$\tilde{E}(x \otimes y) = 2\pi i \sum_{j > k} \theta_{jk} \delta_j(x) \otimes \delta_k(y).$$

Then we see that

$$\frac{d^r}{dt^r} [m_t(x, y)_\alpha] = m_t(\tilde{E}^r(x, y))_\alpha$$

for each  $\alpha \in \mathbb{Z}^n$ . Now  $\tilde{E}^r(x, y)$  is a finite sum of elementary tensors, so the series defining  $m_t(\tilde{E}^r(x, y))$  converges uniformly in  $t$  by the above result. But  $m_t(\tilde{E}^r(x, y))$  is the series of  $r$ -th derivatives of the terms of the series for  $m_t(x, y)$ . So by Lemma A.1, we have

$$\frac{d^r}{dt^r} m_t(x, y) = m_t(\tilde{E}^r(x, y)) \in \mathcal{S}(\mathbb{Z}^n).$$

This shows the deformation is smooth.  $\square$



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